Principles of Simplex Algorithm

$$\text{Max } Z = c_1 x_1 + c_2 x_2 + \cdots\cdots c_n x_n$$

subject to
$$a_{11} x_1 + a_{12} x_2 + \cdots\cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots\cdots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots\cdots + a_{mn} x_n = b_m$$
$$x_1, x_2, \cdots x_n \geq 0$$

Apply a transformation to the above;

$$\left.\begin{aligned} x_1 + \bar{a}_{1,m+1} x_{m+1} + \cdots\cdots + \bar{a}_{1s} x_s + \cdots\cdots + \bar{a}_{1n} x_n &= \bar{b}_1 \\ x_r + a_{r,m+1} x_{m+1} + \cdots\cdots + \bar{a}_{rs} x_s + \cdots\cdots \bar{a}_{rn} x_n &= \bar{b}_r \\ x_m + \bar{a}_{m,m+1} x_{m+1} + \cdots\cdots + \bar{a}_{ms} x_s + \cdots\cdots \bar{a}_{m,n} x_n &= \bar{b}_m \end{aligned}\right\} (S1)$$

The variables $x_1, x_2, \cdots x_m$ which appear with a unit coefficient in only one equation and with zeros in all other equations are called basic variables. The other $n-m$ variables are called non basic variables.

DEFINITIONS

1) A pivot operation is a sequence of elementary row operations that reduces the coefficient of a specified variable to unity in one equation and zero elsewhere.

2) The solution obtained from the transformed system by setting the non-basic variables to zero is called a basic solution. In system (S1), a basic solution is given by
$x_1 = \bar{b}_1 \cdots\cdots x_m = \bar{b}_m, \quad x_{m+1} = 0, \cdots x_n = 0$

If the value of $\bar{b}_1 \cdots \bar{b}_m$ are all non-negative then this basic solution will be called a basic feasible solution.

3) A basic feasible solution is a basic solution in which the values of the basic variables are non-negative. With m constraints and n variables, the maximum number of basic solutions to the standard LP is finite and is given by

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \qquad \text{Ex: } \begin{matrix} m=3 \\ n=5 \end{matrix} \qquad \frac{5!}{3!\,(2)!} = 10$$

Every corner in the solution space is a basic feasible sol'n.

General Steps of the Simplex Algorithm.

1) Start with an initial basic feasible solution (BFS) in standard / canonical form.

2) Improve the solution by finding another BFS with a better objective function value.

3) Continue step 2 until no such improvements are possible.

Suppose an initial basic solution to (S1) is given by

$$\text{Basic}: \quad X_i = \bar{b}_i \geqslant 0 \quad \text{for } i = 1, 2, \dots m$$

$$\text{Non-basic}: \quad X_j = 0 \quad \text{for } j = m+1, \dots n$$

The set of basic variables will be called a basis and will be denoted by X_B.

Let the objective function coefficients of the basic variables will be denoted by C_B. For the initial basis $X_B = (x_1, x_2, \dots X_m)$ and $C_B = (C_1, C_2, \dots C_m)$

Since the non-basic variables are zero the value of the objective function Z corresponding to the initial basic feasible sol'n is given by

$$Z = C_B X_B = C_1 \bar{b}_1 + C_2 \bar{b}_2 + \dots + C_m \bar{b}_m$$

Given this the simplex method first examines whether the present solution is optimal or not. In case if it is not optimal it obtains an adjacent basic feasible solution with a larger value of Z)

An adjacent basic feasible solution differs from the present basic feasible solution in exactly one basic variable.

To obtain an adjacent basic feasible sol'n the simplex method makes one of the basic variables a non-basic variable and brings in a non-basic variable as a basic variable in its place. The problem is to select the appropriate basic and non-basic variables such that an exchange between them will give maximum improvement to the objective function.

This is done by increasing the value of the non-basic variable by one unit and examining the resulting change in the value of the objective function.

To illustrate consider the non basic variable X_s. Let us increase its value from $\emptyset$ to 1 and study its effect on the objective function. Since the values of the other non-basic variables are zero system (S1) can be written as

$$\left.\begin{aligned} X_1 \qquad\qquad + \bar{a}_{1s} X_s &= \bar{b}_1 \\ X_r + \bar{a}_{rs} X_s &= \bar{b}_r \\ X_m + \bar{a}_{ms} X_s &= \bar{b}_m \end{aligned}\right\} \quad (S2)$$

From system (S2), we obtain the new solution when X_s increases from $\emptyset$ to 1.

$$X_i = \bar{b}_i - \bar{a}_{is} \qquad \text{for } i = 1, 2, \ldots m$$

$$X_s = 1$$

$$X_j = 0 \qquad \text{for } j = m+1, \ldots n \text{ and } j \neq s$$

The new value of the objective function is:

$$z = \sum_{i=1}^{m} c_i (\bar{b}_i - \bar{a}_{is}) + c_s$$

Hence, the net change in the value of z per unit increase in X_s, denoted by $\bar{c}_s$ is the following

$$\bar{C}_s = \text{new value of } Z - \text{old value of } Z$$

$$\bar{C}_s = \sum_{i=1}^{m} C_i(\bar{b}_i - \bar{a}_{is}) + C_s - \sum_{i=1}^{m} C_i \bar{b}_i = C_s - \sum_{i=1}^{m} C_i \bar{a}_{is}$$

Where $\bar{C}_s$ is called the relative profit of the nonbasic variable X_s, as opposed to its actual profit C_s in the objective function. If the $\bar{C}_s > 0$ then the objective function Z can be increased further by making X_s a basic variable.

The relative profit coefficient of a nonbasic variable X_j denoted by $\bar{C}_j$ is given by

$$\bar{C}_j = C_j - C_B \bar{P}_j$$

where $\bar{P}_j$ corresponds to j^{th} column in the equation system.

Condition of Optimality

In a maximization problem a basic feasible solution is optimal if the relative profits of its nonbasic variables are all negative or zero.

Suppose $\bar{C}_s = \max \bar{C}_j > 0$ so that initial basic feasible solution is not optimal. We would like to increase X_s as much as possible to get the largest increase in Z. But as X_s increases the values of the basic variables change, and their new values are obtained from system (S2) as;

$$X_i = \bar{b}_i - \bar{a}_{is} X_s \quad \text{for} \quad i = 1, 2, \ldots m$$

If the coefficient of $\bar{a}_{is} < 0$ then X_i increases as X_s is increased.

if $\bar{a}_{is} = 0$ then X_i does not change

if $\bar{a}_{is} > 0$ then X_i decreases as X_s is increased. It may turn negative if X_s is increased indefinetely. Hence the maximum increase in X_s is given by the following formula;

$$\max X_s = \min_{\bar{a}_{is}>0} \left[\frac{\bar{b}_i}{\bar{a}_{is}} \right]$$ Minimum ratio rule.

Let $$\frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\bar{a}_{is}>0} \left[\frac{\bar{b}_i}{\bar{a}_{is}} \right]$$

Hence, when X_s is increased to $\bar{b}_r / \bar{a}_{rs}$ the basic variable X_r turns zero first. Therefore X_r is replaced by X_s in the basis. The new basic feasible solution is given by

$$X_i = \bar{b}_i - \bar{a}_{is} \left(\frac{\bar{b}_r}{\bar{a}_{rs}} \right) \quad \text{for all } i$$

$$X_s = \frac{\bar{b}_r}{\bar{a}_{rs}}$$

$X_j = 0$ for all other basic variables $j = m+1, \ldots n$ and $J \neq S$

Since a unit increase in X_s increases the objective function Z by $\bar{C}_s$ units the total increase in Z is given by;

$$(\bar{C}_s) \left(\frac{\bar{b}_r}{\bar{a}_{rs}} \right) > 0$$