EXAMPLE PROBLEM SET II

Q1) Solve the following problem with the fourth-order RK method:

$$\frac{d^2y}{dx^2} + 0.6\frac{dy}{dx} + 8y = 0$$

where $y(0) = 4$ and $y'(0) = 0$. Solve from $x = 0$ to 5 with $h = 0.5$.
Plot your results.

SOLUTION

Consider the second order differential equation is provided in the text book:

$$\frac{d^2y}{dx^2} + 0.6\frac{dy}{dx} + 8y = 0$$

Now divide system in two parts as

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = -0.6z - 8y$$

Where $y_0 = 4$, $z_0 = 0$, and $x = 0$ to 5.

The Fourth-Order Runge-Kutta method is given as

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

All the slopes at the beginning of the interval, here k_{ij} is the i^{th} value of k for the j^{th} independent variable.

$$\begin{aligned} k_{11} &= f(0,4,0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} k_{12} &= g(0,4,0) \\ &= -0.6\times(0) - 8\times 4 \\ &= -32 \end{aligned}$$

Now,

Calculate the first values of y_1 and y_2 at the midpoint:

$$\begin{aligned} y_{mid} &= y_0 + k_{11} \times \frac{h}{2} \\ &= 4 + 0 \times \frac{0.5}{2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} z_{mid} &= z_0 + k_{12} \times \frac{h}{2} \\ &= 0 + (-32) \times \frac{0.5}{2} \\ &= -8 \end{aligned}$$

The y_{mid} and z_{mid} used to calculate the first set of midpoint slopes.

$$\begin{aligned} k_{21} &= f(0.6, 4, -8) \\ &= -8 \end{aligned}$$

$$\begin{aligned} k_{22} &= g(0.6, 4, -8) \\ &= -0.6 \times (-8) - 8 \times (4) \\ &= \text{-27.2} \end{aligned}$$

These are used to determine the second set of midpoint predictions

$$
\begin{aligned}
y_{mid} &= y_0 + k_{21} \times \frac{h}{2} \\
&= 4 + (-8) \times \frac{0.5}{2} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
z_{mid} &= z_0 + k_{22} \times \frac{h}{2} \\
&= 0 + (-27.2) \times \frac{0.5}{2} \\
&= -6.8
\end{aligned}
$$

These are used to second set of midpoint slopes.

$$
\begin{aligned}
k_{31} &= f(0.75, 1.6, -6.8) \\
&= -6.8
\end{aligned}
$$

$$
\begin{aligned}
k_{32} &= g(0.75, 2, -6.8) \\
&= -0.6 \times (-6.8) - 8 \times (2) \\
&= -11.92
\end{aligned}
$$

There are used to determine the predictions at the end of interval

$$
\begin{aligned}
y_{end} &= y_0 + k_{31} \times h \\
&= 4 + (-6.8) \times 0.5 \\
&= -0.6
\end{aligned}
$$

$$
\begin{aligned}
z_{end} &= z_0 + k_{32} \times h \\
&= 0 + (-11.92) \times 0.5 \\
&= -5.96
\end{aligned}
$$

These are used to determine the end point slopes.

$$
\begin{aligned}
k_{41} &= f(1, 0.6, -5.96) \\
&= -5.96
\end{aligned}
$$

$$
\begin{aligned}
k_{42} &= g(1, 0.6, -5.96) \\
&= -0.6 \times (-5.96) - 8 \times (0.6) \\
&= -1.224
\end{aligned}
$$

Now,

The value of $k_{11}, k_{21,} k_{31}$ and k_{41} are used to determine the value of

Put $i = 0$,

In equation of Fourth Order Runge- Kutta method:

$$
\begin{aligned}
y_1 &= y_0 + \frac{1}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}) \times 0.5 \\
&= 4 + \frac{1}{6}[0 + 2(-8 + -6.8) + -5.96] \times 0.5 \\
&= 1.036666666 \\
&\approx 1.0367
\end{aligned}
$$

Similarly,

The value of $k_{12}, k_{22,} k_{32}$ and k_{42} used to determine the second value

Further solve,

$$
\begin{aligned}
z_1 &= z_0 + \frac{1}{6}(k_{12} + 2k_{22} + 2k_{32} + k_{42}) \times 0.5 \\
&= 0 + \frac{1}{6}[-32 + 2(-27.2 + -11.92) + -1.22] \times 0.5 \\
&= -9.288333 \\
&= -9.28834
\end{aligned}
$$

The next condition will be,

$$
\begin{aligned}
x_1 &= 0.5 \\
y_1 &= 1.0367 \\
z_1 &= -9.29
\end{aligned}
$$

The value of k_{11},

$$
\begin{aligned}
k_{11} &= f(0.5, 1.0367, -9.29) \\
&= -9.29
\end{aligned},
$$

The value of $k_{12,}$

$$
\begin{aligned}
k_{12} &= g(0.5, 1.0367, -9.29) \\
&= -0.6 \times (-9.29) - 8 \times 1.0367 \\
&= 5.574 - 8.2936 \\
&= 2.72
\end{aligned}
$$

Now follow the all process to find all value as above solved.

Calculate midpoint as given below:

$$\begin{aligned} y_{mid} &= y_1 + k_{11} \times \frac{h}{2} \\ &= 1.0367 + (-9.29) \times \frac{0.5}{2} \\ &= -1.2858 \end{aligned}$$

$$\begin{aligned} z_{mid} &= z_1 + k_{12} \times \frac{h}{2} \\ &= -9.29 + (-2.72) \times \frac{0.5}{2} \\ &= -9.97 \end{aligned}$$

Determine the value of k_{21} and k_{22}

$$\begin{aligned} k_{21} &= f(0.5 + 0.5, -1.2858, -9.97) \\ &= f(1, -1.2858, -9.97) \\ &= -9.97 \end{aligned}$$

Determine the value of k_{21} and k_{22}

$$\begin{aligned} k_{21} &= f(0.5 + 0.5, -1.2858, -9.97) \\ &= f(1, -1.2858, -9.97) \\ &= -9.97 \end{aligned}$$

And,

$$\begin{aligned} k_{22} &= g(1, -1.2858, -9.97) \\ &= -0.6 \times (-9.97) - 8 \times (-1.2858) \\ &= 16.2684 \\ &= 16.27 \end{aligned}$$

Calculate the y_{mid} and z_{mid}

$$\begin{aligned} y_{mid} &= y_1 + k_{21} \times \frac{h}{2} \\ &= 1.0367 + (-9.97) \times \frac{0.5}{2} \\ &= -1.4558 \\ &= -1.46 \end{aligned}$$

The value of z_{mid}.

$$\begin{aligned} z_{mid} &= z_1 + k_{22} \times \frac{h}{2} \\ &= -9.29 + (16.27) \times \frac{0.5}{2} \\ &= -5.2225 \end{aligned}$$

Now,

Determine the value of k_{31} and k_{32},

$$\begin{aligned}k_{31} &= f(1.5,-1.46,-5.22)\\ &= -5.22\\ k_{32} &= g(1.5,-1.46,-5.22)\\ &= -0.6\times(-5.22)-8\times(-1.46)\\ &= 14.78\end{aligned}$$

Determine y_{end} and z_{end}

$$\begin{aligned}y_{end} &= y_1 + k_{31}\times h\\ &= 1.0367+(-5.22)\times 0.5\\ &= -1.5733\\ z_{end} &= z_1 + k_{32}\times h\\ &= -9.29+(14.78)\times 0.5\\ &= -1.9\end{aligned}$$

These are used to determine the end point slopes,

$$\begin{aligned}k_{41} &= f(2,-1.5733,-1.9)\\ &= -1.9\\ k_{42} &= g(2,-1.5733,-1.9)\\ &= -0.6\times(-1.9)-8\times(-1.5733)\\ &= 13.74\end{aligned}$$

The value of $k_{11}, k_{21,} k_{31}$ and k_{41} are used to determine the value of

Put,

$i = 1$,

In equation of Fourth Order Runge- Kutta method:

$$\begin{aligned}y_2 &= y_1 + \frac{1}{6}\left[k_{11} + 2\times(k_{21}+k_{31}) + k_{41}\right]\times 0.5\\ &= 1.0367 + \frac{1}{6}\left[-9.29 + 2(-9.97-5.22) + -1.9\right]\times 0.5\\ &= -2.4276\end{aligned}$$

Similarly,

The value of $k_{12}, k_{22,} k_{32}$ and k_{42} used to determine the second value

Further solve,

$$\begin{aligned}z_2 &= z_1 + \frac{1}{6}\left[k_{12} + 2\times(k_{22}+k_{32}) + k_{42}\right]\times 0.5\\ &= -9.29 + \frac{1}{6}\left[-2.72 + 2(16.27+14.78) + 13.74\right]\times 0.5\\ &= -3.1996\end{aligned}$$

For the further iteration, process in the same way;

The result for the 4^{th} order RK method are tabulated and plotted below.

x	y	z	k_{11}	k_{12}	y_{mid}	z_{mid}	k_{21}	k_{22}	y_{mid}	z_{mid}	k_{31}	k_{32}
0	4.0000	0.0000	0.00	-32.00	4.00	-8.00	-8.00	-27.20	2.00	-6.80	-6.80	-11.92
0.5	1.0367	-9.2887	-9.29	-2.72	-1.29	-9.97	-9.97	16.27	-1.46	-5.22	-5.22	14.78
1	-2.4276	-3.1969	-3.20	21.34	-3.23	2.14	2.14	24.53	-1.89	2.94	2.94	13.38
1.5	-1.5571	5.3654	5.37	9.24	-0.22	7.67	7.67	-2.88	0.36	4.65	4.65	-5.68
2	1.1539	4.0719	4.07	-11.67	2.17	1.15	1.15	-18.07	1.44	-0.44	-0.44	-11.27
2.5	1.4810	-2.3334	-2.33	-10.45	0.90	-4.95	-4.95	-4.21	0.24	-3.39	-3.39	0.07
3	-0.2935	-3.6375	-3.64	4.53	-1.20	-2.50	-2.50	11.13	-0.92	-0.86	-0.86	7.87
3.5	-1.1319	0.3723	0.37	8.83	-1.04	2.58	2.58	6.76	-0.49	2.06	2.06	2.66
4	-0.1853	2.6602	2.66	-0.11	0.48	2.63	2.63	-5.42	0.47	1.31	1.31	-4.56
4.5	0.7241	0.6564	0.66	-6.19	0.89	-0.89	-0.89	-6.57	0.50	-0.99	-0.99	-3.42
5	0.3782	-1.6258	-1.63	-2.05	-0.03	-2.14	-2.14	1.51	-0.16	-1.25	-1.25	2.00

Q2) Solve the following problem over the interval from x =0 to 1 using a step size of 0.25 where $y(0)$ =1. Display all your results on the same graph.

$$\frac{dy}{dt} = (1 + 4t)\sqrt{y}$$

(a) Analytically.
(b) Euler's method.
(c) Heun's method without iteration.
(d) Ralston's method.
(e) Fourth-order RK method.

SOLUTION

Consider the initial value problem

$$\frac{dy}{dt} = (1+4t)\sqrt{y},$$

over the interval $t \in [0, 1]$with the initial condition $y(0)=1$ and step size is $h = 0.25$.

(a)

Objective is to solve the problem analytically.

The ordinary differential equation is $\frac{dy}{dt} = (1+4t)\sqrt{y}$. Use the method of separation of variables as;

$$\frac{dy}{\sqrt{y}} = (1+4t)dt.$$

Now, integrate both the sides under the limit $t=0, y=1$ (because $y(0)=1$) to $t=t, y=y$ and get,

$$\int_{y=1}^{y=y} \frac{dy}{\sqrt{y}} = \int_{t=0}^{t=t} (1+4t)dt$$

$$\left[2y^{\frac{1}{2}}\right]_1^y = \left[t+2t^2\right]_0^t$$

$$2\sqrt{y} - 2 = t + 2t^2 - 0$$

$$\sqrt{y} = \frac{2t^2+t+2}{2}$$

And then, $y = \left(\frac{2t^2+t+2}{2}\right)^2$.

Hence, the analytical solution is $y = \left(\frac{2t^2+t+2}{2}\right)^2$.

(b)

Solve the problem using Euler's method.

The formula of Euler's method is defined as:

$y_{i+1} = y_i + y_i' \times h$.

Here $y_i' = (1+4t_i)\sqrt{y_i}$ and $h = 0.25$. For $i = 1, t_i = 0.00$ and $y_i = 1$, then

$$\begin{aligned} y_i' &= (1+4\times 0)\sqrt{1} \\ &= 1 \\ y_{i+1} &= 1 + 1\times 0.25 \\ &= 1.25. \end{aligned}$$

Now, for $i = 2, t_2 = t_1 + h,$ that is, $t_2 = 0.25$.

For next calculation consider the table shown below:

i	$t_i = t_{i-1} + h$	y_i	$y_i' = y_i(t_i^2 - 1.1)$	$y_{i+1} = y_i + y_i' \times h$
2	0.25	1.25	2.236	1.809
3	0.50	1.809	4.035	2.818
4	0.75	2.818	6.715	4.497
5	1.00	4.497		

(c)

Solve the problem using Heun's method without iteration.

The formula of Heun's method is defined as:

$$\begin{aligned} y_{i+1}^P &= y_i + y_i' \times h, \\ y_{i+1}'^P &= (1+4t_{i+1})\sqrt{y_{i+1}^P} \\ y_{i+1} &= y_i + \frac{y_i' + y_{i+1}'^P}{2} \times h. \end{aligned}$$

For $i = 1, t_i = 0.00$ and $y_i = 1$, then

$$\begin{aligned} y_i' &= (1+4\times 0)\sqrt{1} \\ &= 1 \\ y_{i+1}^P &= 1 + 1\times 0.25 \\ &= 1.25 \\ y_{i+1}'^P &= (1+4\times 0.25)\sqrt{1.25} \\ &= 2.236 \\ y_{i+1} &= 1 + \frac{1_i + 2.236}{2} \times 0.25 \\ &= 1.405. \end{aligned}$$

Now, for $i = 2, t_2 = t_1 + h,$ that is, $t_2 = 0.5$.

For next calculation consider the table shown below:

i	t_i	y_i	y'_i	y^P_{i+1}	y'^P_{i+1}	y_{i+1}
2	0.25	1.405	2.370	1.9971	4.239	2.231
3	0.50	2.231	4.481	3.351	7.322	3.706
4	0.75	3.706	7.701	5.631	11.865	6.152
5	1.00	6.152				

(d)

Solve the problem using Ralston's method.

The formula for Ralston's method is defined as:

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right) \times h,$$
$$k_1 = f(t_i, y_i),$$
$$k_2 = f\left(t_i + \frac{3}{4}h,\, y_i + \frac{3}{4}k_1 h\right).$$

For the calculation consider the table shown below:

i	t_i	y_i	k_1	k_2	y_{i+1}
1	0.00	1	1	1.907	1.401
2	0.25	1.401	2.367	3.735	2.221
3	0.50	2.221	4.471	6.559	3.687
4	0.75	3.687	7.680	10.755	6.119
5	1.00	6.119			

(e)

Solve the problem using Fourth order RK method.

The formula for Fourth order RK method is defined as:

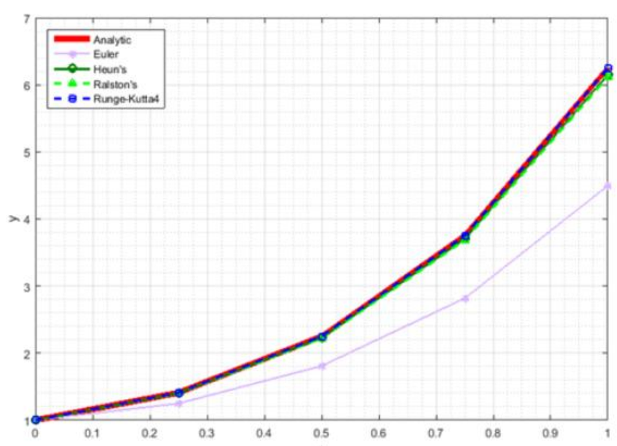
$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = f(t_i, y_i),$$
$$k_2 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right),$$
$$k_3 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right),$$
$$k_4 = f(t_i + h, y_i + k_3 h).$$

For the calculation consider the table shown below:

i	t_i	y_i	k_1	k_2	k_3	k_4	y_{i+1}
1	0	1	1	1.591	1.642	2.375	1.410
2	0.25	1.410	2.375	3.266	3.371	4.503	2.249
3	0.50	2.249	4.499	5.869	6.045	7.758	3.753
4	0.75	3.753	7.749	9.779	10.038	12.5129	6.249
5	1	6.249					

This is the required solution of initial value problem using Fourth order RK method.

The required graph will be:



Q3) Solve the following problem numerically from t =0 to 3:

$$\frac{dy}{dt} = -2y + t^2 \qquad y(0) = 1$$

Use the third-order RK method with a step size of 0.5.

SOLUTION

Consider the initial value problem

$$\frac{dy}{dt} = -2y + t^2,$$

over the interval $t \in [0, 3]$ with the initial condition $y(0) = 1$ and step size is $h = 0.5$.

Objective is to solve the problem numerically by the third-order RK method.

The formula for third-order RK method is defined as:

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 4k_2 + k_3)$$
$$k_1 = f(t_i, y_i),$$
$$k_2 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right),$$
$$k_3 = f(t_i + h, y_i - k_1 h + 2k_2 h).$$

For $i = 1, t_i = 0.00$ and $y_i = 1$, then

$$\begin{aligned} k_1 &= -2\times 1 + 0^2 \\ &= -2 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}(-2)\right) \\ &= f(0.25, 0.5) \\ &= -2\times 0.5 + (0.25)^2 \\ &= -0.9375 \end{aligned}$$

$$\begin{aligned} k_3 &= f(0 + 0.5, 1 - (-2)\times 0.5 + 2\times(-0.9375)\times 0.5) \\ &= f(0.5, 1.0625) \\ &= -2\times 1.0625 + (0.5)^2 \\ &= -1.875 \end{aligned}$$

$$\begin{aligned} y_{i+1} &= 1 + \frac{0.5}{6}(-2 + 4\times(-0.935) - 1.875) \\ &= 0.3645. \end{aligned}$$

Now, for $i = 2, t_2 = t_1 + h,$ that is, $t_2 = 0.5$.

For next calculation consider the table shown below:

i	t_i	y_i	k_1	k_2	k_3	y_{i+1}
2	0.50	0.3645	-0.4791	0.0729	-0.3541	0.3194
3	1.00	0.3194	0.3611	0.7430	0.4861	0.6377
4	1.50	0.6377	0.9745	1.2997	1.0995	1.2438
5	2.00	1.2438	1.5123	1.8186	1.6373	2.1125
6	2.50	2.1125	2.0249	2.3249	2.1499	3.2354
7	3.00	3.2354				

This is the required solution of initial value problem using third order RK method.

Q4) Use **(a)** Euler's and **(b)** the fourth-order RK method to solve

$$\frac{dy}{dt} = -2y + 5e^{-t}$$

$$\frac{dz}{dt} = -\frac{yz^2}{2}$$

over the range $t = 0$ to 0.4 using a step size of 0.1 with $y(0) = 2$ and $z(0) = 4$.

SOLUTION

Consider the problem

$$\frac{dy}{dt} = -2y + 5e^{-t},$$

$$\frac{dz}{dt} = -\frac{yz^2}{2},$$

over the interval $t \in [0, 0.4]$ with the initial condition $y(0) = 2, z(0) = 4$ and step size is $h = 0.1$.

(a)

Objective is to solve the problem using Euler's method.

First solve the equation $\frac{dy}{dt} = -2y + 5e^{-t}$. The formula of Euler's method is defined as:

$y_{i+1} = y_i + y_i' \times h$.

Here $y_i' = -2y_i + 5e^{-t_i}$ and $h = 0.1$.

For $i = 1, t_i = 0.00$ and $y_i = 2$, then

$$\begin{aligned} y_i' &= -2 \times 2 + 5e^{-0} \\ &= 1 \end{aligned}$$

$$\begin{aligned} y_{i+1} &= 2 + 1 \times 0.1 \\ &= 2.1. \end{aligned}$$

Now, for $i = 2, t_2 = t_1 + h,$ that is, $t_2 = 0.1$.

For next calculation consider the table shown below:

i	$t_i = t_{i-1} + h$	y_i	$y_i' = -2y_i + 5e^{-t_i}$	$y_{i+1} = y_i + y_i' \times h$
2	0.1	2.1	0.324	2.1324
3	0.2	2.1324	-0.1712	2.1153
4	0.3	2.1153	-0.5265	2.0626
5	0.4	2.0626		

The formula of Euler's method for $\frac{dz}{dt} = -\frac{yz^2}{2}$ is defined as:

$z_{i+1} = z_i + z_i' \times h$.

Here $z_i' = -\frac{y_i \times z_i^2}{2}$ and $h = 0.1$. The above obtained values of y_i's will be used here.

For $i = 1, t_i = 0.00$ and $z_i = 4$, then

$$
\begin{aligned}
z_i' &= -\frac{2 \times 4^2}{2} \\
&= -16 \\
z_{i+1} &= 4 - 16 \times 0.1 \\
&= 2.4.
\end{aligned}
$$

Now, for $i = 2, t_2 = t_1 + h,$ that is, $t_2 = 0.1$.

For next calculation consider the table shown below:

i	$t_i = t_{i-1} + h$	y_i	z_i	z_i'	$z_{i+1} = z_i + z_i' \times h$
2	0.1	2.1	2.4	-6.048	1.7952
3	0.2	2.1324	1.7952	-3.4361	1.4516
4	0.3	2.1153	1.4516	-2.2286	1.2287
5	0.4	2.0626	1.2287		

(b)

Solve the problem using Fourth order RK method.

The formula for Fourth order RK method is defined as:

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 &= f(t_i, y_i), \\
k_2 &= f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right), \\
k_3 &= f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right), \\
k_4 &= f(t_i + h, y_i + k_3 h).
\end{aligned}
$$

Suppose that

$$f(t, y) = -2y + 5e^{-t},$$

$$g(y, z) = -\frac{yz^2}{2}.$$

$$k_{y1} = f(t_i, y_i),$$

$$k_{y2} = f(t_i + 0.5h,\ y_i + 0.5k_{y1}h),$$

$$k_{y3} = f(t_i + 0.5h,\ y_i + 0.5k_{y2}h),$$

$$k_{y4} = f(t_i + h,\ y_i + k_{y3}h)$$

For $i = 1, t_i = 0.00$ and $y_i = 2, z_i = 4$, then

$$\begin{aligned} k_{y1} &= -2 \times 2 + 5e^{-0} \\ &= 1, \end{aligned}$$

$$\begin{aligned} k_{y2} &= -2 \times 2.05 + 5e^{-0.05} \\ &= 0.656, \end{aligned}$$

$$\begin{aligned} k_{y3} &= -2 \times 2.0328 + 5e^{-0.05} \\ &= 0.6905, \end{aligned}$$

$$\begin{aligned} k_{y4} &= -2 \times 2.069 + 5e^{-0.1} \\ &= 0.3861 \end{aligned}$$

And then

$$\begin{aligned} y_{i+1} &= 2 + \frac{0.1}{6}(1 + 2 \times 0.656 + 2 \times 0.6905 + 0.3861) \\ &= 2.068. \end{aligned}$$

Also,

$$\begin{aligned} k_{z1} &= \frac{-2 \times 4^2}{2} \\ &= -16, \end{aligned}$$

$$\begin{aligned} k_{z2} &= \frac{-2.05 \times 3.2^2}{2} \\ &= -10.496, \end{aligned}$$

$$\begin{aligned} k_{z3} &= \frac{-2.0328 \times 3.475^2}{2} \\ &= -12.275 \end{aligned}$$

$$\begin{aligned} k_{z4} &= \frac{-2.069 \times 2.7725^2}{2} \\ &= -7.952 \end{aligned}$$

And then

$$\begin{aligned} z_{i+1} &= 4 - \frac{0.1}{6}(16 + 2 \times 10.496 + 2 \times 12.275 + 7.952) \\ &= 2.84. \end{aligned}$$

When $i = 2, t_i = 0.1$ and $y_i = 2.068, z_i = 2.84$, then

$k_{y1} = 0.3882,\ k_{z1} = -8.35,$

$k_{y2} = 0.1287,\ k_{z2} = -6.1338,$

$k_{y3} = 0.1547,\ k_{z3} = -6.6657,$

$k_{y4} = -0.0733,\ k_{z4} = -4.93.$

And corresponding $y_{i+1} = 2.0827,\ z_{i+1} = 2.1938$.

When $i = 3, t_i = 0.2$ and $y_i = 2.0827, z_i = 2.1938$, then

$k_{y1} = -0.0717,\ k_{z1} = -5.0117,$

$k_{y2} = -0.2642,\ k_{z2} = -3.9254,$

$k_{y3} = -0.245,\ k_{z3} = -4.1287,$

$k_{y4} = -0.4123,\ k_{z4} = -3.264.$

And corresponding $y_{i+1} = 2.0576,\ z_{i+1} = 1.787$.

When $i = 4, t_i = 0.3$ and $y_i = 2.0576,\ z_i = 1.787$, then

$k_{y1} = -0.4112,\ k_{z1} = -3.287,$

$k_{y2} = -0.5507,\ k_{z2} = -2.683,$

$k_{y3} = -0.5368,\ k_{z3} = -2.7743,$

$k_{y4} = -0.6563,\ k_{z4} = -2.28.$

And corresponding $y_{i+1} = 2.0038,\ z_{i+1} = 1.5126$.

This is the required solution of initial value problem using Fourth order RK method.