ON SOME GENERALIZED EINSTEIN METRIC CONDITIONS

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ABSTRACT

Let (M, g) be a semi-Riemannian manifold. We denote by g, R, S, κ and C the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g), respectively. Using these tensors we can define the $(0, 6)$-tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$ and $Q(A, T)$, where A is a symmetric $(0, 2)$-tensor and T a generalized curvature tensor (see, e.g., [9, Section 1], [11, Section 1] and [14, Sections 1 and 2]).

A semi-Riemannian manifold (M, g), $\dim M = n \geq 2$, is said to be an *Einstein manifold* [2], or an *Einstein space*, if at every point of M its Ricci tensor S is proportional to g, i.e., $S = (\kappa/n)\, g$ on M, assuming that κ is constant when $n = 2$. According to [2, p. 432], this condition is called the *Einstein metric condition*. Einstein manifolds form a natural subclass of several classes of semi-Riemannian manifolds which are determined by curvature conditions imposed on their Ricci tensor [2, Table, pp. 432-433]. These conditions are called *generalized Einstein metric conditions* [2, Chapter XVI].

The tensor $R \cdot C - C \cdot R$ of every semi-Riemannian Einstein manifold (M, g), $n \geq 4$, satisfies the following identities [12, Theorem 3.1] (see also [8, p. 100001-1] and [14, p. 107])

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n}\, Q(g, R) = \frac{\kappa}{(n-1)n}\, Q(g, C) = \frac{1}{n-1}\, Q(S, R) = \frac{1}{n-1}\, Q(S, C),$$

$$R \cdot C - C \cdot R = \frac{\kappa}{n-1}\, Q(g, C) - Q(S, C) = Q(\frac{\kappa}{n-1}\, g - S, C). \quad (1)$$

We can express the tensor $R \cdot C - C \cdot R$ of some non-Einstein and non-conformally flat semi-Riemannian manifolds (M, g), $\dim M \geq 4$, as a linear combination of $(0, 6)$-Tachibana tensors $Q(A, T)$, e.g., $A = g$ or $A = S$ and $T = R$ or $T = C$. These conditions form a family of generalized Einstein metric conditions. Semi-Riemannian manifolds, and in particular hypersurfaces isometrically immersed in spaces of constant curvature, satisfying such conditions were investigated in several papers, see, e.g., [1, 4, 7, 9, 10, 11, 13, 14]. We refer to [5] (see also [8]) for a survey of results on manifolds (hypersurfaces) satisfying such conditions.

Below we present selected results related to our talk.

If a non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M, g), $n \geq 4$, satisfies the following two curvature conditions of pseudosymmetry type: $R \cdot R = L_1\, Q(g, R)$ and $C \cdot C = L_2\, Q(g, C)$, where L_1 and L_2 are some functions, then the curvature tensor R is a linear combination of the Kulkarni-Nomizu products formed by the metric tensor g and the Ricci tensor S [15, Theorem

3.1, Teorem 3.2 (ii)]. A non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M, g), $n \geq 4$, with curvature tensor R expressed by the above-mentioned linear combination of the Kulkarni-Nomizu products is called a *Roter type manifold*, or a *Roter manifold*, or a *Roter space* (see, e.g., [3, Section 15.5], [5, 9, 11], [14, Section 4]). Every Roter space satisfies (1) (see, e.g., [9, Proposition 3.3], [11, Theorem 2.4 (ii)]).

Let M, $\dim M \geq 4$, be a hypersurface isometrically immersed in a space of constant curvature such that at every point M has exactly two distinct principal curatures, λ_1 with multiplicity p and λ_2 with multiplicity $n - p$, $2 \leq p \leq n - 2$. If $(p_1 - 1)\lambda_1 + (n - p - 1)\lambda_2 \neq 0$, then M is a Roter space, and in a consequence (1) holds on M [6, Theorem 3.3].

Let M, $\dim M \geq 4$, be a hypersurface isometrically immersed in a space of constant curvature such that at every point M has exactly three distinct principal curatures. If the condition $R \cdot C - C \cdot R = Q(g, T)$, where T is a generalized curvature tensor, is satisfied on M, then the tensor T is a linear combination of the curvature tensor R and Kulkarni-Nomizu products formed by the metric tensor g, the Ricci tensor S and its square S^2 (cf. [10, Theorem 5.2]).

If at every point of a non-quasi-Einstein and non-conformally flat hypersurface M, $\dim M \geq 4$, isometrically immersed in a semi-Riemannian space of constant curvature the tensor $R \cdot C - C \cdot R$ is a linear combination of the tensors $Q(g, C)$ and $Q(S, C)$, then (1) holds on M [11, Theorem 5.4].

Keywords Einstein manifold · Einstein generalized metric condition · pseudosymmetric type curvature condition · Roter space · hypersurface

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