Cauchy-Riemann Geometry[1]: An Introduction to the Main Problems

Sorin Dragomir[2]

The ordinary Cauchy-Riemann system $\overline{\partial} f = 0$ on $\mathbb{C}^n$ $(n \geq 2)$ induces on every real hypersurface $M \subset \mathbb{C}^n$ the *tangential Cauchy-Riemann equations*

$$\overline{\partial}_b u = 0 \tag{1}$$

(a first order overdetermined PDE system, with variable C^∞ coefficients) and every C^1 solution $u : M \to \mathbb{C}$ to (1) is a *CR function* on M. A CR structure is a recast of (1) as an involutive complex distribution $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank $n-1$, and the restriction to M of a holomorphic function (on a neighborhood $\Omega \supset M$) is a solution to (1). The *CR extension* problem is whether a point $x_0 \in M$ admits a neighborhood $\Omega \subset \mathbb{C}^n$ such that the restriction morphism $\mathcal{O}(\Omega) \to \mathrm{CR}^1(U)$ is an epimorphism (with $U = \Omega \cap M$). Given an *abstract* CR structure $T_{1,0}(M)$ on a real $(2n-1)$-dimensional manifold (not necessarily embedded into $\mathbb{C}^n$) the *CR embedding* problem is whether a point $x_0 \in M$ admits a neighborhood $U \subset M$ and a CR immersion $\Psi : U \to \mathbb{C}^n$ [so that the portion of $T_{1,0}(M)$ over U is actually induced by the complex structure of the ambient space $\mathbb{C}^n$]. We review results (old and new) on the two fundamental problems mentioned above, with an emphasis on the differential geometric objects needed in their study (cf. [3] and [1]), and indicate their relationship to mathematical physics (cf. [7], [8], [5], and [6]).

References

[1] E. Barletta & S. Dragomir & K.L. Duggal, *Foliations in Cauchy-Riemann geometry*, Mathematical Surveys and Monographs, Vol. 140, American Mathematical Society, 2007.

[2] *Selected topics in Cauchy-Riemann Geometry*, Edited by S. Dragomir, Quaderni di Matematica, Napoli, Vol. 9, 2001.

[3] S. Dragomir & G. Tomassini, *Differential Geometry and Analysis on CR Manifolds*, Progress in Mathematics, Vol. 246, Birkhäuser, Boston-Basel-Berlin, 2006.

---

[1]Lecture delivered at the *International Geometry Symposium in Memory of Prof. Erdogan Esin*, Gazi University, Faculty of Science, 9-10 February 2023.

[2]Università degli Studi della Basilicata, Dipartimento di Matematica, Informatica ed Economia, Potenza, Italy, e-mail: `sorin.dragomir@unibas.it`

[4] *Geometry of Cauchy-Riemann submanifolds*, Edited by S. Dragomir & M.H. Shahid & F.R. Al-Solamy, Springer Science, Singapore, 2016.

[5] C.D. Hill & J. Lewandowski & P. Nurowski, *Einstein's equations and the embedding of* 3*-dimensional CR manifolds*, `arXiv:0709.3660v2 [math.DG] 22 May 2008`

[6] R. Penrose, *Physical space-time and nonrealizable CR structures*, Bull. A.M.S., N.S., 8(1983), 427-448.

[7] I. Robinson, *A solution of the Maxwell-Einstein equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys., 7(1959), 351-352.

[8] I. Robinson & A. Trautman, *Cauchy-Riemann structures in optical geometry*, 4th Marcel Grossmann Meeting on the Recent Developments of General Relativity, 1985.