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A New Iterative Method for Flow Calculation in Intake and Exhaust Systems of Internal Combustion Engines

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Abstract—In this paper, a new iterative method to find analytical approximate solution with *a priori* error bounds of Euler equations is constructed. This method is an improvement of the one presented in [1] and lets using the temporal domain get a bigger Chebyshev polynomial to approach the initial value in each step.

Keywords—Approximate solution, Cauchy-Kovalevskaya method, Chebyshev polynomial, MacCormack algorithm.

1. INTRODUCTION: ONE-DIMENSIONAL TRAVELLING WAVES

As outlined in previous work [1] about travelling waves, in the particular case of sound waves it is possible to assume the amplitude of oscillations to be small. The result was that the equations of motion were linear and were easily solved. A particular solution of these equations is any function of $x \pm at$ corresponding to a travelling wave whose profile moves with velocity a, its shape remaining unchanged; by the profile of a wave, the distribution of density, velocity, etc. is understood, along the direction of propagation. Since the velocity u, the density ρ, and the pressure p (and the other quantities) in such a wave are functions of the same quantity $x \pm at$, they can be expressed as functions of one another, in which the coordinates and time do not explicitly appear ($\rho = \rho(p), u = u(p)$, and so on).

When the wave amplitude is not necessarily small, these simple relations do not hold [2]. It is found, however, that a general solution of the exact equation of motion can be obtained, in the form of a travelling plane wave which is a generalization of the solution $f(x + at)$ of the approximate equations valid for small amplitudes. To derive this solution, one begins from the

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requirement that, for a wave with any amplitude, the velocity can be expressed as a function of the density.

In the absence of shock waves the flow is adiabatic. If the gas is homogeneous at some initial instant, then $s = constant$ at all times, and assuming this in what follows, the pressure is thus a function of the density only.

In a plane sound wave propagated in the x-direction, all quantities depend only on x and t (see [3]). Following Euler equations, one gets

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u \\ p + \rho u^2\end{bmatrix} = \vec{0}. \tag{1}$$

The general relation between the velocity and the density or the pressure in the wave can be obtained as follows [4]:

$$u = \pm \int \frac{a}{\rho} d\rho = \pm \int \frac{dp}{\rho a}, \tag{2}$$

whence,

$$x = t[u \pm a(u)] + f(u), \tag{3}$$

where $f(u)$ is an arbitrary function of the velocity, $a(u)$ is given by (2) and (3) determines the velocity (and therefore, all other quantities) as an implicit function of x and t, i.e., the wave profile at every instant. For any given value of u, we have $x = t[k1] + k2$, i.e., the point where the velocity has a given value moves with constant velocity; in this sense, the solution obtained is a travelling wave. The two signs in (3) correspond to waves propagated (relative to the gas) in the positive and negative x-directions. The flow described by equations (2) and (3) is often called a *simple wave*, and is essentially different from the one obtained in the limiting case of small amplitudes. The velocity of a point in the wave profile is

$$v = u \pm a. \tag{4}$$

It may be conveniently regarded as a superposition of the propagation of a disturbance relative to the gas with the velocity of sound and the movement of the gas itself with velocity u. The velocity u is now a function of the density, and therefore, is nonconstant in the profile. Thus, in the general case of a plane wave with arbitrary amplitude, there is no definitive constant "wave velocity". Since the velocities of different points in the wave profile are different, the profile changes its shape in the course of time.

If it is considered a wave propagated in the positive x-direction, for which $v = u + a$, the velocity of propagation of a given point in the wave profile increases with the density [4]. If it is denoted by a_0 the velocity of sound for a density equal to the equilibrium density ρ_0, then in compressions $\rho > \rho_0$ and $a > a_0$, while in rarefactions $\rho < \rho_0$ and $a < a_0$.

The inequality of the velocity of different points in the wave profile causes its shape to change in the course of time: the points of compression move forward and those of rarefaction are left behind. An example can be observed in Figure 1, where the evolution of the pressure pulse in a straight duct, with a constant diameter of 50 mm has been represented. In the lower scale, time for the generated wave can be read and in the upper scale, time for the deformed wave. This evolution has been calculated through solving equation (3) with a high-resolution numerical method called CE-SE [5]. The duct has been densely meshed and the two calculation points have been separated with 28 m between them.

Finally, the profile may become such that a discontinuity (shock-wave) is formed and the function is one-valued everywhere except at the discontinuity itself. The wave profile then has the form shown in Figure 2, where a deformed wave measured in an experimental facility can be observed.

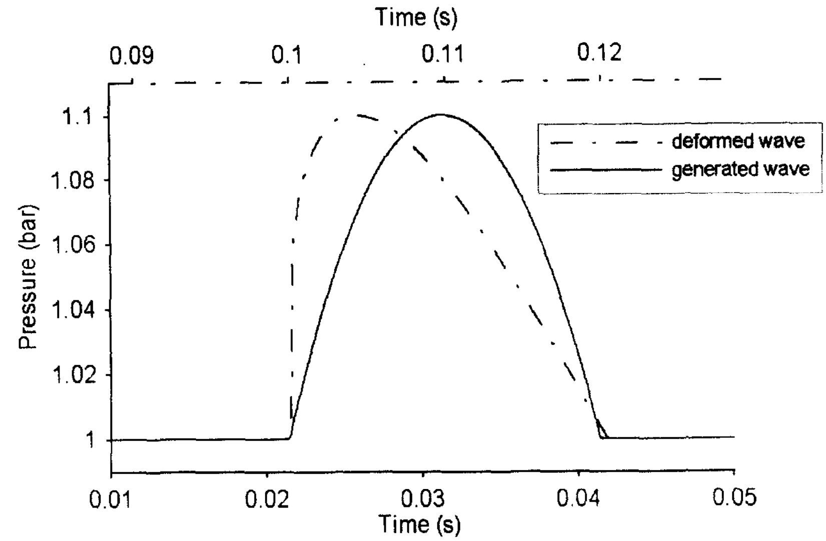

Figure 1. Numeric solution of a pressure evolution after travelling through a straight duct.

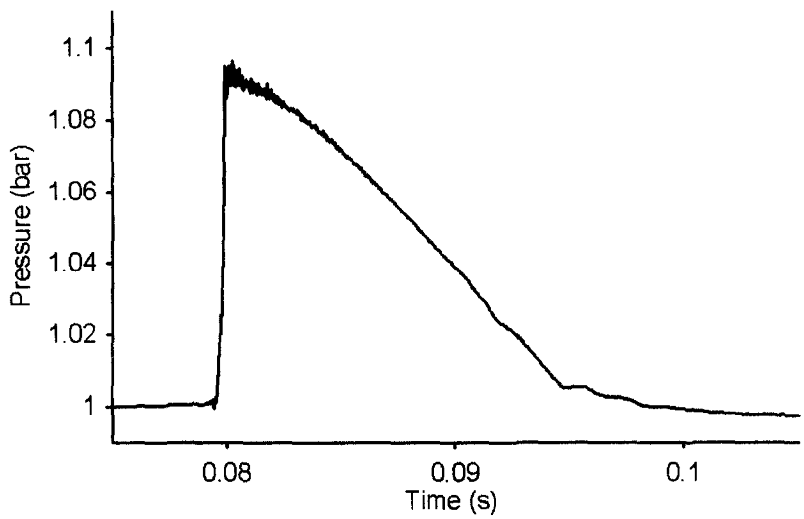

Figure 2. Discontinuity measured in a compression wave after travelling through a straight duct.

When the discontinuities are formed, the wave ceases to be a simple wave. The cause of this can be briefly stated thus: when surfaces of discontinuity are present, the wave is reflected from them, and therefore, ceases to be a wave travelling in one direction.

The presence of discontinuities (shock waves) results in the dissipation of energy. The formation of discontinuities therefore leads to a marked damping of the wave, which becomes the main interest of its correct prediction.

It is clear from the above that discontinuities must ultimately be formed in every simple wave which contains regions where the density decreases in the direction of propagation. The only case where discontinuities do not occur is a wave in which the density everywhere increases monotonically in the direction of propagation.

The wave is no longer a simple one when a discontinuity has been formed, the time and place of formation on the discontinuity can be determined analytically. As a consequence, if the nonlinearity of these problems is to be found, the approximate solution has to be defined for enough time far from the origin of the time interval. With this objective, this new iterative model is introduced. This lets temporal domain solution get bigger in the method of one step which was presented in [1], because its temporal domain does not arrive sufficiently far to show whole nonlinearity. The new idea is to obtain a polynomial approximate solution in a fixed space which will be a new initial value for each iteration.

The paper is organised as follows. In the second section, the iterative method description is presented. Moreover, the boundary conditions are also described. In the third section, the analytical-numerical solution method based on the Cauchy-Kovalevskaya theorem and Frobenius algorithm and the intermediate step based in the polynomial Chebyshev approximate is explained. As these appear to be very conservative, the fourth section is dedicated to a comparison between the solution obtained and numerical solutions well validated in the literature. Finally, some concluding remarks are presented.

2. NEW ITERATIVE MODEL

The simplification of energy equations of Euler equations and further application of Cauchy-Kovalevskaya method addresses to the equation [1]

$$\frac{\partial}{\partial x}\begin{bmatrix}\rho\\ u\end{bmatrix} = \frac{1}{u^2-\gamma(p_0/\rho_0^\gamma)\rho^{\gamma-1}}\begin{pmatrix}-u & \rho\\ \gamma\dfrac{p_0}{\rho_0^\gamma}\rho^{\gamma-2} & -u\end{pmatrix}\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ u\end{bmatrix}. \tag{5}$$

This problem requires that $(u^2-\gamma(p_0/\rho_0^\gamma)\rho^{\gamma-1})$ is different from zero, but if this term were zero the physical problem is not an analytical solution. So the problem is well defined.

As pressure is always positive, the following change of variable may be used:

$$y = \rho^{\gamma-1}$$

and substitution into (5) yields

$$\frac{\partial}{\partial x}\begin{bmatrix}y\\ u\end{bmatrix} = \frac{1}{u^2-\gamma(p_0/\rho_0^\gamma)y}\begin{pmatrix}-u & (\gamma-1)y\\ \dfrac{\gamma p_0}{(\gamma-1)\rho_0^\gamma} & -u\end{pmatrix}\frac{\partial}{\partial t}\begin{bmatrix}y\\ u\end{bmatrix}. \tag{6}$$

This system is equivalent to (5), but it has a more simple computational treatment.

In this article, two boundary values are studied. The first imposes a synodical function in the pressure. This, in the variables of problem (6), has the following expression:

$$\begin{aligned} y(0,t) &= \rho_0^{\gamma-1}\left(1+\frac{A}{p_0}\sin(kt)\right)^{(\gamma-1)/\gamma}, \\ u(0,t) &= \frac{2c_0}{\gamma-1}\left(\left(1+\frac{A}{p_0}\sin(kt)\right)^{(\gamma-1)/2\gamma}-1\right), \end{aligned} \tag{7}$$

where A is the amplitude of the wave considered and k the frequency. The constants p_0, ρ_0, and c_0 are the pressure, reference density, and sound speed, respectively, and γ adiabatic constant.

The second condition is when a synodical function in the velocity is imposed. This, in the variables of problem (6), has the following expression:

$$\begin{aligned} y(0,t) &= \rho_0^{\gamma-1}\left(1+\frac{\gamma-1}{2c_0}A\sin(kt)\right)^2, \\ u(0,t) &= A\sin(kt). \end{aligned} \tag{8}$$

This system with these boundary conditions produces no linearity result when high amplitudes are worked (see [1]). To finalize the presentation of the problem, the domain where the problem has the solution, is

$$D = \{t\in[0,+\infty[,\ x\in\mathbb{R}\}.$$

Although, as the solution is periodic to space variable, it is only necessary to study one period.

Once the system has been presented, the method iterative which obtained the analytical solution almost everywhere is described.

This method has the following steps.

1. The first is to divide the period in two parts and transfer until the half point coincides with the origin. This allows us to study two cases: the increase and decrease part of the period.
2. Each part is studied using the Cauchy-Kovalevskaya method. And, for each fixed spacial point, two break (a, b) points are selected where the solution is defined into the interval $[a, b]$, because the solutions can only be calculated on a domain (see [1]).
3. The solutions in the two subdomains are transferred and joined in various periods until the solutions are defined on an interval $[-c, c]$.
4. Afterwards, this new solution is approximated by a polynomial using the Chebyshev polynomial approximation method.
5. Finally, this new approximation solution is used as a boundary value for the next iteration.

Once the method has been presented, the analysis is carried out. First, the theoretical index and domains *a priori* of each part are presented. Afterwards, the numerical analysis is presented, in it the computational results are presented for both boundary conditions, as are criteria to find the break points.

3. *A PRIORI* ESTIMATIONS OF THEORETICAL DOMAINS AND DIMENSIONS

In order to develop an *a priori* study of dimensions and domains, each step is treated separately. First, the index truncation to the approximate solution which are obtained to approximate the problem solution for the Cauchy-Kovalevskaya method plus algorithm Frobenius with a correct boundary value (see [1]) are studied and afterwards, the approximate for Chebyshev polynomial of joint solutions.

3.1. Construction of a Polynomial Approximation

In this section, the theoretical development of the method and the theoretical error of the solution are summarized, (details in [1]).

The problem is

$$\frac{\partial}{\partial x}\begin{pmatrix} y \\ u \end{pmatrix} = \frac{1}{u^2 - A_1 y}\begin{pmatrix} -u & A_2 y \\ A_3 & -u \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} y \\ u \end{pmatrix}, \tag{9}$$

where

$$A_1 = \gamma\frac{P_0}{\rho_0^{\gamma}}, \qquad A_2 = \gamma - 1, \qquad A_3 = \frac{A_1}{A_2}. \tag{10}$$

Thus, according to the Cauchy-Kovalevskaya theorem (see [6,7]), system (9) admits a local analytical solution of the form

$$y(x,t) = \sum_{n\geq 0}\sum_{m\geq 0} y_{nm}x^n t^m, \qquad u(x,t) = \sum_{n\geq 0}\sum_{m\geq 0} u_{nm}x^n t^m. \tag{11}$$

Now, the coefficients y_{nm}, u_{nm} are calculated from the series of boundary conditions

$$\begin{aligned} y(0,t) &= \sum_{m\geq 0} y_{0m}t^m, \\ u(0,t) &= \sum_{m\geq 0} u_{0m}t^m. \end{aligned} \tag{12}$$

By equating the coefficients of t^m of the series above, we obtain

$$y_{0m}, \quad u_{0m}, \qquad m \geq 0.$$

The rest of y_{nm}, u_{nm} variables are obtained by a recursive method, using the equations obtained from (9)

$$\begin{aligned} \frac{\partial y}{\partial x} &= -h(x,t)u(x,t)\frac{\partial y}{\partial t} + A_2 h(x,t)y(x,t)\frac{\partial u}{\partial t}, \\ \frac{\partial u}{\partial x} &= A_3 h(x,t)\frac{\partial y}{\partial t} - h(x,t)u(x,t)\frac{\partial u}{\partial t}, \end{aligned} \tag{13}$$

where

$$h(x,t) = \frac{1}{u^2(x,t) - A_1 y(x,t)}.$$

Following the Frobenius method, substituting (11) in (13) and equating the coefficients of the series obtained on both sides of the equation, the coefficients y_{nm}, u_{nm}, $n \geq 1$, $m \geq 1$ may be calculated recursively, starting from y_{0m}, u_{0m} as given by (12).

Next, the calculation of the convergence domain of the theoretical solution obtained is presented. Considering the constant $y_0 \geq 0$ and

$$y = y_0 + z,$$

and substituting in (9) yields the following system:

$$\frac{\partial}{\partial x}\begin{pmatrix} z \\ u \end{pmatrix} = \frac{1}{u^2 - A_1 z - A_1 y_0}\begin{pmatrix} -u & A_2(z+y_0) \\ A_3 & -u \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} z \\ u \end{pmatrix}, \tag{14}$$

and it is an analytical function in $|t| + |z| + |u| \leq r$, taking

$$0 < r < \min\left(-\frac{A_1}{2} + \sqrt{\left(\frac{A_1}{2}\right)^2 + A_1 y_0},\ |y_0|\right). \tag{15}$$

According to the method presented in [8], the following expressions may be defined:

$$\begin{aligned} M_1 &= \frac{1}{A_1 y_0 - (r^2 + A_1 r)} \max\{r, A_2(r+y_0), A_3\}, \\ M_\Phi &= \max\left\{\sup_{|t|\leq r}\{y(0,t)\},\ \sup_{|t|\leq r}\{u(0,t)\}\right\}, \\ \tilde{M}_2 &= 2M_1 M_\Phi, \qquad \tilde{M} = \max\left\{M_1, \tilde{M}_2\right\}. \end{aligned} \tag{16}$$

Choosing

$$0 < s < \frac{r}{\sqrt{3}}, \qquad \alpha = 2\tilde{M} + 1 + \sqrt{5\tilde{M}}, \qquad \tilde{R} = \min\left\{\alpha s, \frac{(1-\alpha)^2 s}{2\tilde{M}}\right\}, \tag{17}$$

the solution converges in

$$|x| \leq \tilde{R}, \qquad |t| \leq \tilde{R}. \tag{18}$$

Now, the truncation index n_2 that will give an admissible error $\varepsilon > 0$ is obtained by considering the truncated solution

$$y(x,t,n_2) = \sum_{n=0}^{n_2}\sum_{m=0}^{n_2} y_{nm}x^n t^m, \tag{19}$$

$$u(x,t,n_2) = \sum_{n=0}^{n_2}\sum_{m=0}^{n_2} u_{nm}x^n t^m, \tag{20}$$

this satisfies

$$\left\| \begin{pmatrix} y(x,t) \\ u(x,t) \end{pmatrix} - \begin{pmatrix} y(x,t,n_2) \\ u(x,t,n_2) \end{pmatrix} \right\|_2 < \varepsilon. \tag{21}$$

Taking

$$\tilde{K} = s + \tilde{R} + \sqrt{s + \tilde{R} + 2s\tilde{R}\tilde{M}} \tag{22}$$

and choosing R_0, such that $0 < R_0 < \tilde{R}$, allows us to define N_1

$$N_1 = \frac{4\tilde{K}R_0\left(\tilde{R} + 2R_0\right)}{\left(\tilde{R} - R_0\right)^2}. \tag{23}$$

Given $\varepsilon > 0$, n_0 and n_1 are taken as the first positive integers satisfying

$$n_0 > \frac{\varepsilon/2\sqrt{2}N_1}{2\left(2R_0\Big/\left(\tilde{R} + R_0\right)\right)}, \tag{24}$$

$$n_1 + 1 > \frac{(\varepsilon/2\psi)\left(\tilde{R} - R_0\right)\Big/\left(\tilde{R} + R_0\right)}{2R_0\Big/\left(\tilde{R} + R_0\right)}, \tag{25}$$

respectively, where

$$\psi \geq \sup_{|t| \leq r} \left\{ \left\| \begin{pmatrix} y(0,t) \\ u(0,t) \end{pmatrix} \right\|_2 \right\}. \tag{26}$$

By taking

$$n_2 = \max\{n_0, n_1\} \tag{27}$$

and calculating $y(x,t,n_2)$, and $u(x,t,n_2)$ using (19) and (20), (21) is verified according to [8].

3.2. Approximation to Initial Value

To study the intermediate step, first some previous theorems are presented for the following development.

PROPOSITION 3.1. *Let $f \in C^\infty([a, x_n])$ and $g \in C^\infty([x_{n+1}, b])$ with $x_n < x_{n+1}$. There is a function $H \in C^k([a,b])$ for $k \geq 5$ satisfying*

$$H = \begin{cases} H(x) = f(x), & x \in [a, x_n], \\ H(x) = g(x), & x \in [x_{n+1}, b]. \end{cases} \tag{28}$$

PROOF. As both functions are sufficiently smooth, the points $f(x_n)$, $f'(x_n)$, $f''(x_n), \dots, f^k(x_n)$, $g(x_{n+1})$, $g'(x_{n+1})$, $g''(x_{n+1}), \dots, g^k(x_{n+1})$ exist.

With these points, a polynomial function, $p \in P([x_n, x_{n+1}])$ can be calculated which coincides with f on the point x_n and its derivatives until order k. The same would happen with g.

With this polynomial function, the function

$$H = \begin{cases} f(x), & x \in [a, x_n], \\ p(x), & x \in [x_n, x_{n+1}], \\ g(x), & x \in [x_{n+1}, b], \end{cases}$$

is $C^k([a,b])$ and satisfies (28).

THEOREM 3.1. *Let a function $f \in C^4([-a,a])$ then it has a finite approximation Chebyshev series (which is defined as $P_n f$ where n is the Chebyshev polynomial dimension, which also coincides with the worked number points) and this approximation satisfies that*

$$\|f - P_n f\|_\infty < \frac{2\, a^{3/2}}{\sqrt{n}n\,(n-1)\,(n-2)} \left\|f^{iv}\right\|_2 .$$

PROOF. The demonstration idea can be found in [9]. Here complete demonstration when $f \in C^4([-a,a]))$ and error annotation when the functions are defined on $[-a,a]$, are shown. This demonstration has two parts. First, the function will be defined on $[-1,1]$ and second, the function will be defined on $[-a,a]$ general.

Let $g \in C^4([-1,1])$ and its Chebyshev approximation $P_\infty g = a_0 + \sum_{k\geq 1} a_k T_k(t)$, where $T_k(t)$ are generated by

$$\begin{aligned} T_0(t) &= 1, \\ T_1(t) &= t, \\ T_k(t) &= 2tT_{k-1}(t) - T_{k-2}(t), \end{aligned}$$

and

$$a_k = \frac{\sqrt{2}}{\pi e_k} \int_0^\pi g(\cos(\theta)) \cos(n\theta)\, d\theta$$

with

$$e_k = \begin{cases} \sqrt{2}, & k = 1, \\ 1, & k > 1. \end{cases}$$

Then this series converges uniformly to the function and beside the Parseval's equality is satisfied,

$$\|g\|_2 = \sum_{k\geq 0} a_k^2. \tag{29}$$

Let g' its differential, as the function is $C^1([-1,1])$, its development in Chebyshev series is able to be calculated (which is defined as $P_\infty g' = a'_0 + \sum_{k\geq 1} a'_k T_k(t)$). Then it is known that the coefficients $a_0, a_1, a_2 \dots$, are related with a'_0, a'_1, a'_2 in the following way:

$$a_n = \frac{1}{2n}\left(e_{n-1} a'_{n-1} - a'_{n+1}\right), \qquad a'_n = \sum_{p\geq 0} (n + 2p - 1)\, a^2_{n+2p-1}. \tag{30}$$

Once these series developments have been defined, they are followed by the study of the truncation index for a fixed error.

Let $f \in C^1([-1,1])$ and $P_n f = a_0 + \sum_{k=1}^n a_k T_k(t)$, then

$$(I - P_n)\, f\,(t) = \sqrt{2} \sum_{k=n+1}^{\infty} a_k T_k(t). \tag{31}$$

Now, using (31), T_k definition, (30) and Schwartz' inequality

$$\begin{aligned} \left|(I - P_n)\, f\,(t)\right|^2 &= 2\left(\sum_{k=n+1}^{\infty} |a_k|\right)^2 = 2\left(\sum_{k=n+1}^{\infty} \frac{\left|e_{k-1} a'_{k-1} - a'_{k+1}\right|}{2n}\right)^2 \\ &\leq \frac{1}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \sum_{k=n+1}^{\infty} \left|e_{k-1} a'_{k-1} - a'_{k+1}\right|^2 \\ &\leq \frac{1}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^2}\, 2 \sum_{k=n+1}^{\infty} \left|e_{k-1} a'_{k-1}\right|^2 + \left|a'_{k+1}\right|^2 \\ &\leq 2 \sum_{k=n+1}^{\infty} \frac{1}{k^2} \sum_{k=n}^{\infty} \left|a'_k\right|^2 . \end{aligned}$$

Thus,

$$\|(I-P_n)f\|_\infty \le \sqrt{2}\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right)^{1/2}\left(\sum_{k=n}^{\infty}|a'_k|^2\right)^{1/2}$$
$$\le \frac{\sqrt{2}}{\sqrt{n}}\|(I-P_{n-1})f'\|_2 .$$

With this, the ∞-norm is connected with norm 2 of its derivatives. If a better index truncation is wanted, the norm 2 has to be connected with norm 2 of its derivatives.

Let $f \in C^1([-1,1])$ and $P_n f = a_0 + \sum_{k=1}^{n} a_k T_k(t)$, then

$$\|(I-P_n)f\|_2^2 = \sum_{k=n+1}^{\infty}|a_k|^2 = \sum_{k=n+1}^{\infty}\frac{|e_{k-1}a'_{k-1}-a'_{k+1}|^2}{4k^2}$$
$$\le \frac{1}{4}\frac{1}{(n+1)^2}\sum_{k=n+1}^{\infty}|e_{k-1}a'_{k-1}-a'_{k+1}|^2$$
$$\le \frac{1}{(n+1)^2}\sum_{k=n}^{\infty}|a'_k|^2$$
$$\le \frac{1}{(n+1)^2}\|(I-P_{n-1})f'\|_2^2 .$$

If $f \in C^4([-1,1])$, joining all previous calculation

$$\|(I-P_n)f\|_\infty \le \frac{\sqrt{2}}{\sqrt{n}}\|(I-P_{n-1})f'\|_2$$
$$\le \frac{\sqrt{2}}{\sqrt{n}n(n-1)(n-2)}\left\|(I-P_{n-4})f^{iv}\right\|_2 .$$

With these calculus, the case $[-1,1]$ is finished. Now, the general case will be studied.

Let $g \in C^4([-a,a])$ and take f defined by $f(t) = g(at)$, for $t \in [-1,1]$. The function f satisfies

$$\|f\|_\infty = \|g\|_\infty, \qquad \|f^n\|_2 = a^{(n-1)/2}\|g^n\|_2 .$$

If $P_n g(x) = P_n f(x/a) = a_0 + \sum_{k=1}^{n} a_k T_k(x/a)$, then

$$\|(I-P_n)g\|_\infty = \|(I-P_n)f\|_\infty$$
$$\le \frac{\sqrt{2}}{\sqrt{n}n(n-1)(n-2)}\left\|(I-P_{n-4})f^{iv}\right\|_2$$
$$= \frac{\sqrt{2}a^{3/2}}{\sqrt{n}n(n-1)(n-2)}\left\|(I-P_{n-4})g^{iv}\right\|_2^2$$

is satisfied.

With this previous calculus, the intermediate step will be developed.

Let an error ε, first, the problem is divided in two parts and the theoretical dimension for an error smaller than $\varepsilon/4$ is calculated by the Cauchy-Kovalevskaya method. These two solutions, both defined on $[-b,b]$, are translated and joined in one. This new function is $C^\infty([-a,a])$ almost everywhere and on the joints points, the error is smaller than $\varepsilon/2$.

As both function are continuous, in each discontinuity point x_d, two point $x_a < x_d < x_p$ can be calculated, each point is defined on one of function and the difference between its value is less than $3\varepsilon/4$. Now, a function, belonging to $C^4([-a,a])$, can be found for the first proposition and this new function can be approximated by a Chebyshev polynomial of dimension n for an error less than ε. With this new boundary value the new iteration can be calculated.

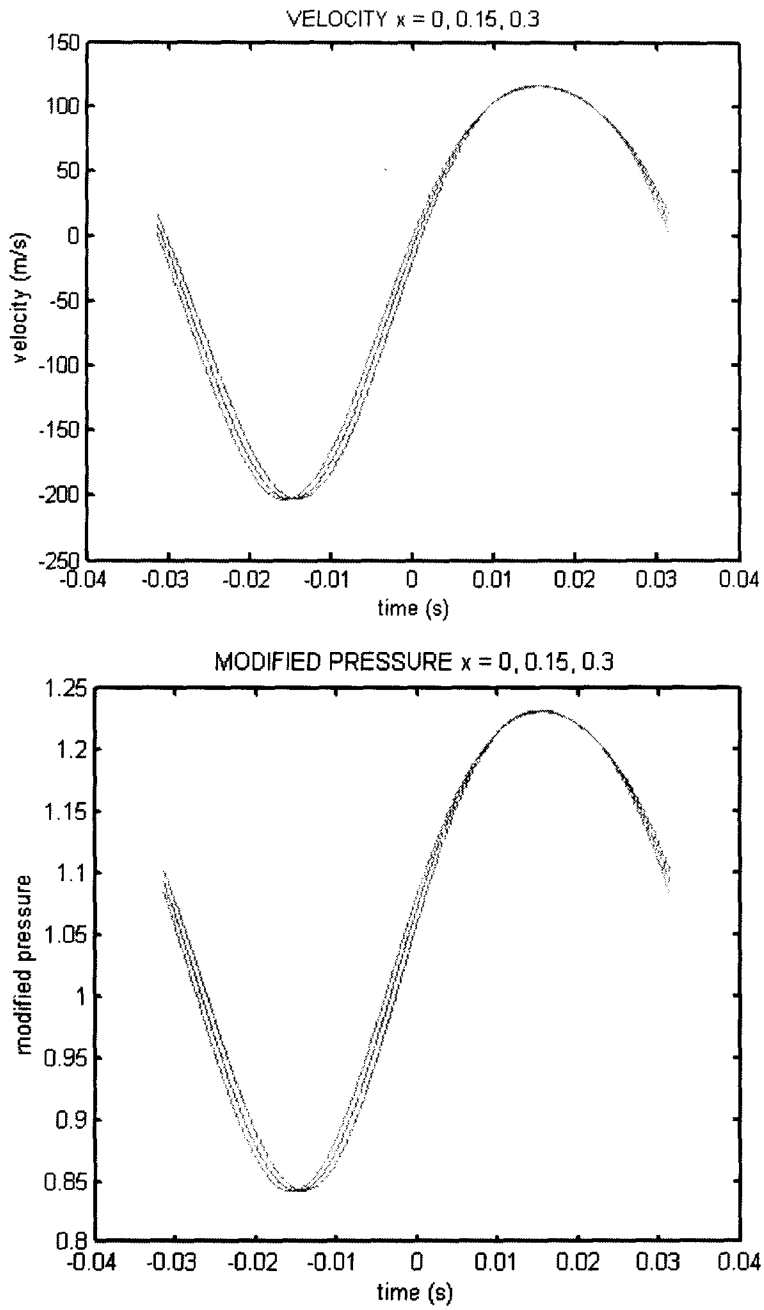

Figure 3. Iterative method case (a).

From this solution, the theoretical index of truncation and domains are very big. However, in the next section, it is shown that these theoretical results are better in reality.

4. COMPARISON WITH THE NUMERICAL MODEL

In this section, the formulated model will be analysed with the numerical model Macpulso developed by the "Departamento de Máquina y Motores Térmicos" of Universidad Politécnica de Valencia. This model works with the McCormack numeric algorithm of resolution conservation laws (see [10]).

In this case (as can be seen in the previous article [1]), the theoretical Cauchy-Kovalevscaya index truncations are very conservative and this problem has better bounds. Also, it can seen that the computational cost increases the exponential form with the dimension, so work with high

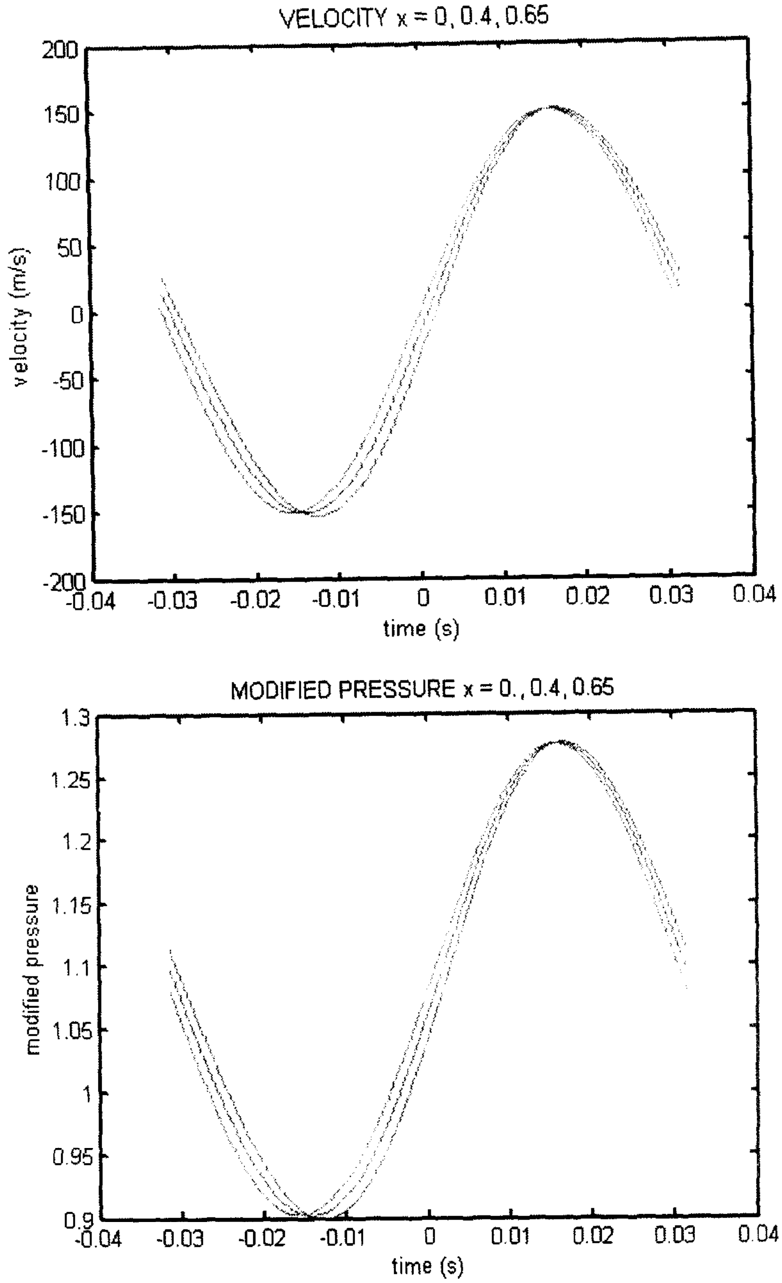

Figure 4. Iterative method case (b).

dimensions is impossible. In this example, it can seen that the results are good if dimension 60 is used and its computational cost are not too much high.

Remembering that the system is

$$\frac{\partial}{\partial x}\begin{bmatrix} y \\ u \end{bmatrix} = \frac{1}{u^2 - \gamma(p_0/\rho_0^\gamma)y}\begin{pmatrix} -u & (\gamma-1)y \\ \dfrac{\gamma p_0}{(\gamma-1)\rho_0^\gamma} & -u \end{pmatrix}\frac{\partial}{\partial t}\begin{bmatrix} y \\ u \end{bmatrix}$$

and the boundary conditions

$$\begin{aligned} y(0,t) &= \rho_0^{\gamma-1}\left(1 + \frac{A}{p_0}\sin(kt)\right)^{(\gamma-1)/\gamma}, \\ u(0,t) &= \frac{2c_0}{\gamma-1}\left(\left(1 + \frac{A}{p_0}\sin(kt)\right)^{(\gamma-1)/2\gamma} - 1\right) \end{aligned} \qquad (32)$$

or

$$
\begin{aligned}
y(0,t) &= \rho_0^{\gamma-1}\left(1+\frac{\gamma-1}{2c_0}A\sin(kt)\right)^2, \\
u(0,t) &= A\sin(kt),
\end{aligned} \tag{33}
$$

where k is the frequency (in this case 100) and A is the amplitude of the wave considered (in the first case 60000 p and in the second case 150 m/s).

In Table 1, the break point in one step and iterative method to error 10^{-2}, error for last step and the evolution of size of domain for each step are presented.

Table 1.

Name	Amp.	Break Point	Error	Size of Domain
casop (a)	60000 p	0.015/0.3	5.1524 10^{-3}	4 of 0.015 and 24 of 0.01
casov (b)	150 m/s	0.2/0.65	1.3548 10^{-3}	1 of 0.2, 3 of 0.1, and 3 of 0.05

In Table 1, a and b show the boundary conditions which are studied (32) and (33), respectively. The error is the maximum, for a fixed space, between the solution and the numeric solution for temporal mesh divided by the numeric solution. And the break points are the limit of previous iteration in the first case and a little previous in the second case.

As can be seen in Table 1, working in the same error, the break points decrease when the amplitude increases. This happens because the errors depend on the initial value and its derivatives norm, and these increase with the amplitude. This aspect is important in the election of break points. Also, it can be seen that the break point decreases in each step. This happens because the solution has to approximate by Chebyshev polynomial.

To finish, a drawn space time of one period when $x = 0.3$ and $x = 0.65$ to case b and $x = 0.15$ and $x = 0.3$ to case a will be shown in Figures 3 and 4. In both cases, it can be seen that the method is able to capture the nonlinearity of the wave.

5. CONCLUSIONS

Summarizing, in this paper, a new iterative method has been presented that allows one to pick up the nonlinearity, besides obtaining an approximation polynomial solution with a given error. Also, a solution algorithm has been presented.

From the point of numerical view, the experimental study shows that the theoretical convergence domain and the theoretical truncation index are very conservative because they are defined by the worst possible case. This does not allow a view of break point election. But as the computational evolution is fast enough, the break point was able to be analysed without being compared to the numerical method in each iteration, which is not beneficial. Another aspect is that the Cauchy-Kovalevskaya method depends more on the approximation in the whole domain (see [1]), this is the reason why the Chebyshev polynomial has been used to approximate the new boundary value in each iteration.

A problem is that the method cannot iterate infinitely, because the error in each iteration depends on previous iteration and the sum of these errors make the initial value, for an iterative, not to be sufficiently correct.

To finish, this analytical solution which has been obtained, will allow a theoretical study of the evolution of smooth wave to be made.

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