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A Collocation Method to Compute One-Dimensional Flow Models in Intake and Exhaust Systems of Internal Combustion Engines

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Abstract—In this paper, an analytical approximate solution of Euler equations is obtained by a collocation method. The collocation method uses two variables Chebyshev polynomial basis and collocation coefficients are obtained solving polynomial equations by means of homotopy methods.

Keywords—Boundary condition, Euler equations, Chebyshev collocation, Homotopy continuation.

1. INTRODUCTION

A fundamental part in wave action models to internal combustion engine design is the flow calculus in ducts. This flow is assumed one-dimensional and it depends on two variables, time and only one space variable. With this simplification, the solution of the problem is the solution of a nonlinear differential system. In the absence of outside forces, the flow is supposed homoentropic and the system can be written as [1]

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u \\ p + \rho u\end{bmatrix} = \vec{0} \tag{1}$$

and

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{2}$$

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with p_0 and ρ_0 constants, the reference pressure and density for each case. In order to work with boundary conditions and apply the Cauchy-Kovalevskaya theorem, the problem requires that [1,2]

$$u^2 - \gamma \frac{p_0}{\rho_0^\gamma} \rho^{\gamma-1} \neq 0, \tag{3}$$

and if this term is zero the physical problem has not an analytical solution. Also, in [1,2], two different type of boundary conditions are studied. In both cases, its value has the following generic expression:

$$\begin{aligned} \rho(0,t) &= g_1(\sin(t)), \\ u(0,t) &= g_2(\sin(t)). \end{aligned} \tag{4}$$

Joining (1) and (4), the general expression of this problem is

$$\begin{aligned} w_t + A(w)\, w_x &= 0, \\ w(0,t) &= g(t). \end{aligned} \tag{5}$$

The existence of analytical solution when the function $A^{-1}(w)$ and g are analytical (as in Euler equations) can be proved by Cauchy-Kovalevskaya theorem [3,4]. Besides, an algorithm which shows the series development of the solution with *a priori* error, was presented in [5]. Based on this algorithm, an iterative algorithm was presented in [2]. This algorithm allows to extend the domain of the solution of the basic algorithm without increasing the dimension of the polynomial solution. However, the principal problem of this method is the high dimension of the polynomial approximate solution.

In this article, a new alternative method based in polynomial collocation is presented. With this method, the solution is obtained as the sum of an initial solution and a polynomial solution with smaller dimension. However, this method needs the boundary over three of its four sides.

In the following section, the problem is generalized to the model

$$\begin{gathered} u_t + A(u)\, u_x = 0, \\ u(x,-1) = f(x), \qquad u(x,1) = h(x), \\ u(-1,t) = g(t), \end{gathered} \tag{6}$$

where A is a polynomial matrix on u, in order to apply the method.

2. GENERALIZED PROBLEM

The Euler equations, when homoentropic flux is studied, are the equations (1). Calculating $\frac{\partial p}{\partial x}$ in (2) and substituting in (1), an expression for the variables $\frac{\partial \rho}{\partial t}$ and $\frac{\partial u}{\partial t}$ is obtained as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ u\end{bmatrix} + \begin{pmatrix} u & \rho \\ \gamma \frac{p_0}{\rho_0^\gamma}\rho^{\gamma-2} & u \end{pmatrix} \frac{\partial}{\partial x}\begin{bmatrix}\rho \\ u\end{bmatrix} = \vec{0}. \tag{7}$$

In the equation, the third matrix coefficient is a noninteger power of density variable. In order to apply the polynomial resolution method, it is convenient to make a change of variable. The following change of variable is used here,

$$T = \frac{p_0}{R\rho_0}\rho^{\gamma-1}, \tag{8}$$

where R is a constant. This variable change can be used because the variable ρ is always positive [1,2]. This variable T has a physical mean, since it is the temperature flux [6].

Taking into account this change and replacing (8) into the equation (7) yields

$$\frac{\partial}{\partial t}\begin{bmatrix}T \\ u\end{bmatrix} = \begin{pmatrix} -u & -(\gamma-1)\,T \\ -\frac{R\gamma}{(\gamma-1)} & -u \end{pmatrix} \frac{\partial}{\partial x}\begin{bmatrix}T \\ u\end{bmatrix}. \tag{9}$$

In this case, the matrix is linear and in general form, system (9) can be written as

$$v_t + A(v)\, v_x = 0, \tag{10}$$

where $A(v)$ is a polynomial matrix function in v. Once the polynomial system has been calculated and since only one boundary condition (4) is defined, new boundary equations have to be defined to be able to apply collocation method. These new conditions will be physical conditions and these are defined on the equation characteristic of the system. These characteristics are defined [7] as the solution of following system,

$$\begin{vmatrix} 1 & 0 & u & (\gamma-1)T \\ 0 & 1 & \dfrac{R\gamma}{(\gamma-1)} & u \\ dt & 0 & dx & 0 \\ 0 & dt & 0 & dx \end{vmatrix} = 0. \tag{11}$$

The physical solution is defined at the points where the boundary is $v_0 = [T_0, 0]$ and in this case, problem (11) is

$$\begin{vmatrix} 1 & 0 & 0 & (\gamma-1)T_0 \\ 0 & 1 & \dfrac{R\gamma}{(\gamma-1)} & 0 \\ dt & 0 & dx & 0 \\ 0 & dt & 0 & dx \end{vmatrix} = 0. \tag{12}$$

Their solutions are

$$x = c_0\,(t_0 \pm t) = x_0 \pm c_0 t, \tag{13}$$

with

$$c_0 = \sqrt{\gamma R T_0}, \tag{14}$$

and t_0 is one of the instants (because the function is periodic) where $g(t_0) = v_0$.

On the characteristic, the solution is constant and its value is v_0, so the problem solution is obtained

$$T\,(c_0\,(s+t_0)\,, s+t_0) = T_0, \qquad u\,(c_0\,(s+t_0)\,, s+t_0) = 0. \tag{15}$$

If equation (10) is combined with boundary condition (4) and characteristic condition (15), being the points 0 and t_0 two instants where the boundary condition is v_0, the problem is written as

$$\begin{gathered} v_t + A(v)\, v_x = 0, \\ v\,(c_0 s, s) = v_0, \qquad v\,(c_0\,(s+t_0)\,, s+t_0) = v_0, \\ v\,(0,t) = g\,(t)\,, \end{gathered} \tag{16}$$

Considering the transformation,

$$\begin{aligned} x &= \frac{2x}{t_0} - 1, \\ z &= \frac{2\left(t - \frac{x}{c_0}\right)}{t_0} - 1, \end{aligned} \tag{17}$$

problem (16) becomes

$$\begin{gathered} v_z - \frac{1}{c_0} A(v)\, v_z + A(v)\, v_x = 0, \\ v\,(-1, z) = g\,(z)\,, \\ v\,(x,-1) = v\,(x,1) = v_0, \end{gathered} \tag{18}$$

and its domain is

$$\{-1 \leq z \leq 1,\ x \geq -1\}.$$

The paper is organized as follows. In Section 3, the collocation with Chebyshev polynomial is applied to the general problem and a nonlinear algebraic polynomial system is obtained. In Section 4, an homotopy continuation method will be used to solve the algebraic problem. In Section 5, the collocation method will be applied to a Burgers equation with known solution. This example is important because the Burgers equation has the same kind of nonlinearity than Euler equations, and its error has been studied. To finish, the method will be applied to Euler equations and a conclusion will be presented (Sections 6 and 7).

3. CHEBYSHEV POLYNOMIAL COLLOCATION

Once the problem has been presented, the Chebyshev polynomial collocation will be studied. The collocation methods have been applied to linear and nonlinear problems where its nonlinearity depends on variables t and x [8–10] with good results. In [11], a linear model based in Chebyshev polynomial collocation has been applied to a linear transformation of Navier-Stokes tridimensional problem. In this article, the Chebyshev polynomial collocation will be applied directly.

The generic form of system (18) is

$$\begin{gathered} u_t + A(u)\, u_x = 0, \\ u(x,-1) = f(x), \quad u(x,1) = h(x), \\ u(-1,t) = g(t), \end{gathered} \tag{19}$$

with $u = (u_1, \ldots, u_N)$ a vector of $\mathbb{R}^N$, $A(u)$ a polynomial matrix function of $\mathbb{R}^{N\times N}$ and f, g, and h vectors of $\mathbb{R}^N$.

The Chebyshev polynomial basis is [12]

$$T_n(x) = \cos(n \arccos(x)), \qquad 0 \leq n. \tag{20}$$

With these polynomials, the collocation basis is defined as

$$B_{ij}(x,t) = \Phi_i(x)\, \Psi_j(t), \qquad 0 \leq i, j, \tag{21}$$

with

$$\Phi_n(x) = (1+x)\, T_n(x), \qquad \Psi_n(t) = \left(1 - t^2\right) T_n(t). \tag{22}$$

It is easy to see that this basis is 0 on the boundary, i.e.,

$$B_{ij}(-1,t) = B_{ij}(1,t) = B_{ij}(x,-1) = 0. \tag{23}$$

With this basis, the collocation function is defined as

$$\tilde{u}_{nm} = \delta(x,t) + \sum_{i,j=1}^{n,m} \alpha_{ij} B_{ij}(x,t), \tag{24}$$

where

$$\delta(x,t) = \frac{1-t}{2}\left(f(x) - f(-1)\right) + \frac{1+t}{2}\left(h(x) - h(-1)\right) + g(t). \tag{25}$$

Function (24) carries out the boundary and initial conditions. Once the function has been chosen, the collocation points are taken as

$$(x_i, t_j) = \left(\cos\frac{\pi}{i}, \cos\frac{\pi}{j}\right), \tag{26}$$

for $1 \le i \le n$ and $1 \le j \le m$. At the points where one of its components is -1 (and the other is named as s) the following results are easily computed

$$B_{ij}(x_i, t_j) = 0, \qquad \frac{\partial B_{ij}}{\partial s}(x_i, t_j) = 0. \tag{27}$$

The algebraic system is obtained by replacing this basis and points of collocation in (19).

A particular case occurs when the collocation is applied to the point $(-1,-1)$ because all $\alpha_{i,j}$ disappear for (27) and the equation does not give any information. So, this has to be changed. In this case, the equation has been replaced with its partial derivatives in x. Thus, the new equation is written as

$$\tilde{u}_{tx} + \tilde{u}_x^T \frac{\partial A(\tilde{u})_1}{\partial \tilde{u}} \tilde{u}_x + A(\tilde{u})\tilde{u}_{xx} = 0, \tag{28}$$

where $\frac{\partial A(\tilde{u})_1}{\partial \tilde{u}}$ is the Jacobian matrix of the first row of the matrix A, i.e., if i is the i^{th} row of A, then,

$$\frac{\partial A(u)_i}{\partial u} = J\left(A(u)_i'\right) = \begin{pmatrix} \frac{\partial A_{i1}}{\partial u_1} & \frac{\partial A_{i1}}{\partial u_2} & \cdots & \frac{\partial A_{i1}}{\partial u_n} \\ \frac{\partial A_{i2}}{\partial u_1} & \frac{\partial A_{i2}}{\partial u_2} & \cdots & \frac{\partial A_{i2}}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial A_{in}}{\partial u_1} & \frac{\partial A_{in}}{\partial u_2} & \cdots & \frac{\partial A_{in}}{\partial u_n} \end{pmatrix}. \tag{29}$$

So, equation (28) calculated on $(-1,-1)$ is

$$\left(\delta_{tx} + \sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial^2 B_{ij}}{\partial x \partial t}\right) = 0. \tag{30}$$

The term $\frac{\partial^2 B_{ij}(-1,-1)}{\partial x \partial t}$ can be calculated in (21), obtaining

$$\frac{\partial B_{ij}^2(-1,-1)}{\partial x \partial t} = 2(-1)^{i+j}, \tag{31}$$

then, equation (28) on the point $(-1,-1)$ is

$$2 \sum_{i,j=1}^{n,m} \alpha_{ij} (-1)^{i+j} = -\delta_{tx}. \tag{32}$$

The other equations are obtained by substituting the points (26) in each equation of (19) as

$$\begin{gathered} \left(\delta_t + \sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial B_{ij}(x_i, t_j)}{\partial t}\right) \\ + A\left(\delta + \sum_{i,j=1}^{n,m} \alpha_{ij} B_{ij}(x_i, t_j)\right)\left(\delta_x + \sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial B_{ij}(x_i, t_j)}{\partial x}\right) = 0. \end{gathered} \tag{33}$$

In the boundary points ($(-1, s)$ or $(s,-1)$), equation (24) is $\tilde{u}(x_i, t_j) = \delta(x_i, t_j)$ by (27), and equation (33) is

$$\left(\delta_t + \sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial B_{ij}(x_i, t_j)}{\partial t}\right) + A(\delta)\left(\delta_x + \sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial B_{ij}(x_i, t_j)}{\partial x}\right) = 0, \tag{34}$$

which is linear.

Summarizing, with the collocation method, a polynomial algebraic system has to be solved. This system is composed by equations (32)–(34). Moreover, the polynomial degree in the boundary points is one and in the other equations, their degree is

$$gr\left(p_{ij}\right) \leq \max\left(gr\left(D\right)+1, gr\left(A\right)+1\right). \tag{35}$$

To conclude this section, note that the problem,

$$\begin{gathered} u_t + A\left(u\right)u_x = 0, \\ u\left(-1,t\right) = f\left(t\right), \quad u\left(1,t\right) = h\left(t\right), \\ u\left(x,-1\right) = g\left(x\right), \end{gathered} \tag{36}$$

can be studied as (19). Its development is the same if basis (22) is changed by

$$\Phi_n\left(x\right) = \left(1-x^2\right)T_n\left(x\right), \qquad \Psi_n\left(t\right) = \left(1+t\right)T_n\left(t\right), \tag{37}$$

and initial function (25) by

$$\delta\left(x,t\right) = \frac{1-x}{2}\left(f\left(t\right)-f\left(-1\right)\right) + \frac{1+x}{2}\left(h\left(t\right)-h\left(-1\right)\right) + g\left(x\right). \tag{38}$$

4. POLYNOMIAL SYSTEM RESOLUTION

In order to solve the polynomial algebraic system, an homotopy continuation method is going to be used [13–15]. This kind of resolution searches all solutions isolated, reals, and complex, of a polynomial system. The main advantage of these methods is that an initial near solution has not to be known, in contrast with to the classical nonlinear methods. Its main disadvantage is that, if a particular solution is wanted to be known, the method cannot be forced to know it because it searches all solutions.

The general algorithm of this kind of methods is: if $P(x) = (p_1(x), p_2(x), \dots, p_n(x)) = 0$ is a polynomial system with degree n, the classical continuation homotopy searches a new polynomial system, $Q(x) = (q_1(x), q_2(x), \dots, q_n(x)) = 0$, which is very easy to solve and, with this new system a homotopy, which joins the easy system with the system $P(x) = 0$, is created. The system $Q(x)$, *initial system*, generates a *good homotopy* $H(x,t)$ if it holds the following three properties [16].

(1) (Triviality) The solutions of $Q(x) = 0$ are known.
(2) (Smoothness) The solution set of $H(x,t) = 0$ for $0 \leq t \leq 1$ consists of a finite number of smooth paths, each parametrized by $t \in [0,1)$.
(3) (Accessibility) Every isolated solution of $H(x,1) = 0$ can be reached by some path originating at $t = 0$. It follows that this path starts at a solution of $H(x,0) = Q(x) = 0$.

In order to generate the initial value, the *degree matrix* $M_D \in \mathbb{N}^{n\times m}$ is defined as the matrix that in the position (i,j) is the sum of the degree in the polynomial p_i, for $i = 1, 2, \dots, n$ of the variables which belong to Z_j, with Z_j an element of the partition set $Z = \{Z_1, Z_2, \dots, Z_m\}$ with dimension

$$k_j = \dim\left\{Z_j\right\} \tag{39}$$

for each element. With this partition, a possible election of the initial system is

$$q_i\left(x\right) = \prod_{j=1}^{n}\left(\prod_{l_1=1}^{d_{i,j}}\left(\sum_{l_2=1}^{k_j} c_{ijl_1l_2}x_{jl_2} + e_{ijl_1}\right)\right), \qquad i = 1, 2, \dots, n. \tag{40}$$

Where the terms $c_{ijl_1l_2}$ and e_{ijl_1} are random in $\mathbb{C} - \{0\}$, k_j is defined in (39) and $d_{i,j}$ is the term in the position (i,j) of matrix M_D. All definitions can be found in [16].

The number of solutions is limited to the Bezout number of M_D for a Z partition [16] which is the number of the different solutions of initial system $Q(x)$. Furthermore, an initial case (40) generates a good homotopy for linear-convex homotopy [17]

$$H(x,t) = (1-t)\,Q(x) + tP(x). \tag{41}$$

To finish, the following algorithm shows a resolution method of homotopy [14,15].

Homotopy continuation algorithm.

$H(x,t), x^*$ initial homotopy solution of $H(x^*,0)=0$ max_iter, min_step, max_step, $\varepsilon > 0$	Inputs
$x^* : \|H(x^*,1)\| < \varepsilon$	Outputs
$t=0$, $h = initial\_step$ $t=h$, $x^*\_previous = x^*$, $x^*\_old = x^*$	Initialization
while $t < 1$ and $h > min\_step$	Algorithm
$x^* = x^* + h(x^* - x^*\_previous)$	Secant Predictor
$Newton(H(x,t), x^*, \varepsilon, max\_iter)$	Newton Corrector
if convergence	
$h = \min(max\_step, c(h))$	Increasing h
$t = \min(1, t+h)$	New time
$x^*\_previous = x^*\_old$	New previous space
else	
$h = \frac{h}{2}$	Decreasing h
$t = t - h$	New time
end if	
end while	

(42)

This algorithm solves the homotopy with a predictor-corrector algorithm. The predictor is the secant method and the corrector is the Newton method because it has a quadratic convergence velocity when a near solution initial has been calculated. In the development, it is very important the election of the homotopy step (in the algorithm is named as h) because if this is too big or small, it can produce way joining or some way can diverge and some solution can be lost.

In the studied case, the step $c(h)$ has been used as: if the previous iteration has converged, in the next iteration, the new step is the previous one multiplied for a constant bigger than one (kh). This constant depends on the problem dimension (if the dimension increase, the constant is smaller). In other articles, there are more complex step controls for more complex polynomial systems [13].

5. BURGERS EQUATION APPLICATION

In this section, Burgers equation with initial value or boundary value is treated. In the first case, the initial solution is equal to the problem solution. This example shows that the collocation method finds the solution and the continuation homotopy method finds the algebraic solution when the dimension increases. The second example is a case where the difference between the initial and the real solution is not polynomial. With this example the convergence to the real solution has been studied when the dimension increases.

5.1. Example 1

The first problem is

$$\begin{aligned} u_t + u u_x &= 0, \\ u(x,-1) &= x, \end{aligned} \tag{43}$$

with domain

$$D = [-1, +\infty[\times [-1,1]. \tag{44}$$

The solution of this problem is known and it is defined as

$$u(x,t) = \frac{x}{2+t}. \tag{45}$$

In this example, the collocation will be applied to the domain

$$D_1 = [-1,1] \times [-1,1].$$

In order to apply collocation method (36), two new boundary conditions have to be defined. In the example, it is defined over the points $(-1,t)$ and $(1,t)$. So, first, the characteristics have been defined, following [3], as the solutions of the differential equation

$$\begin{aligned} \frac{dx}{dt} &= x_0, \\ x(-1) &= x_0. \end{aligned} \tag{46}$$

The solution of this problem is $x = x_0(t+2)$, and the solution of (43) is constant on the characteristic $u = x_0$. Now, taking $x_0 = 1$, the boundary $(-1,t)$ has been calculated as

$$u(-1,t) = \frac{-1}{t+2}. \tag{47}$$

In the same way, the boundary $(1,t)$ has been defined as

$$u(1,t) = \frac{1}{t+2}. \tag{48}$$

Joining equations (47),(48) with problem (43), the collocation method can be applied to

$$\begin{aligned} u_t + u u_x &= 0, \\ u(x,-1) &= x, \\ u(-1,t) = \frac{-1}{t+2}, &\qquad u(1,t) = \frac{1}{t+2}, \end{aligned} \tag{49}$$

and (49) has the same solution as (43).

Once the collocation problem has been defined, with basis (37), collocation points (26), and initial function (38)

$$\delta(1,t) = \frac{x}{t+2}, \tag{50}$$

the method is applied. Clearly, the initial solution is equal to problem solution (49).

Applying the collocation method, the algebraic system is for the point $(-1,-1)$,

$$2 \sum_{i,j=1}^{n,m} \alpha_{ij} (-1)^{i+j} = 0, \tag{51}$$

by (32); and for the other points, equation set (33) can be expressed as

$$\begin{aligned}\sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial B_{ij}(x_i,t_j)}{\partial t} + \delta_x \sum_{i,j=1}^{n,m} \alpha_{ij} B_{ij}(x_i,t_j) + \delta \sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial B_{ij}(x_i,t_j)}{\partial x} \\ + \left(\sum_{i,j=1}^{n,m} \alpha_{ij} B_{ij}(x_i,t_j) \right) \left(\sum_{i,j=1}^{n,m} \alpha_{ij} \frac{\partial B_{ij}(x_i,t_j)}{\partial x} \right) = 0,\end{aligned} \tag{52}$$

where the independent terms have been put in the right side and after simplified.

When system (51),(52) is studied, all equations have information. Another important aspect is that the solution $\alpha_{ij} = 0$, for $1 \le i \le n$ and $1 \le j \le m$, is a solution of the algebraic system.

If the degree of the algebraic system is calculated, in the boundary points, it is 1, while in the internal points, its degree is 2. So, there are $n + m - 1$ linear equations and $(n-1)(m-1)$ quadratic equations.

Once the system is calculated, the homotopy continuation method has to be applied to a partition with one set with all α_{ij}. In this case, if the equations are ordered by rows as in the drawing,

$$\begin{array}{cccc} \bullet\!\rightarrow & \bullet\!\rightarrow & \cdots & \bullet\!\rightarrow \\ & \cdots & \cdots & \\ \bullet\!\rightarrow & \bullet\!\rightarrow & \cdots & \bullet\!\rightarrow \\ \nwarrow & \leftarrow & \longleftarrow & \\ \bullet\!\rightarrow & \bullet\!\rightarrow & \cdots & \bullet\nwarrow \end{array} \tag{53}$$

the Bezout number of its degree matrix is $2^{(n-1)(m-1)}$. With these elements and the generated initial solution for the definition (40), which is solved with a typical algorithm LU for linear systems [18], the algorithm (42) obtains the system solutions.

In Table 1, the maximum error of the coefficients α_{ij} of several dimensions is presented.

Table 1. Example 1 errors.

dim	$n = m = 3$	$n = m = 4$	$n = m = 5$	$n = m = 6$
max error	2.2058 10^{-17}	3.5632 10^{-16}	4.5510 10^{-16}	1.561 10^{-17}

So, the homotopy approximates with accuracy to the real solutions.

5.2. Example 2

The second problem is

$$\begin{aligned} u_t + u u_x &= 0, \\ u(-1,t) &= \frac{-1}{t+2}, \end{aligned} \tag{54}$$

with domain $D = [-1,1] \times]-1, +\infty[$ and its solution is $u(x,t) = x/(2+t)$.

In order to apply the collocation method over the domain $D_1 = [-1,1] \times [-1,1]$, the solution over $(x,-1)$ and $(x,1)$ must be defined. With the same arguments as in the Example 1, these functions are

$$u(x,-1) = x, \qquad u(x,1) = \frac{x}{3}, \tag{55}$$

and the Burgers problem is defined as

$$\begin{aligned} u_t + u u_x &= 0, \\ u(x,-1) = x, \qquad u(x,1) &= \frac{x}{3}, \\ u(-1,t) &= \frac{-1}{t+2}. \end{aligned} \tag{56}$$

Once the problem is defined, and with basis (37), collocations points (26), and initial function (38),

$$\delta(1,t) = \frac{(2-t)(x+1)}{3} - \frac{1}{t+2}, \tag{57}$$

the method can be applied. In this case, the initial and real solution are very different and its difference is not a polynomial. Applying the method, the algebraic system is on the point $(-1,-1)$,

$$2\sum_{i,j=1}^{n,m} \alpha_{ij}(-1)^{i+j} = \frac{-4}{3}, \tag{58}$$

by (32), and on the other points, equation set (33). When the system is studied, all equations have information. If the degree of the algebraic system is calculated, on the boundary points the degree is 1, while in the internal points, it is 2. So, there are $n+m-1$ linear equations and $(n-1)(m-1)$ quadratic equations.

Once the system is calculated, the homotopy continuation method has to be applied for a partition containing one set with all α_{ij}. In this case, the Bezout number of its degree matrix is $2^{(n-1)(m-1)}$. With these elements, the system solution is obtained.

In Table 2, the maximum error is presented for different domain and a mesh of 0.00002 of side. There are two different types of error: near the boundary (-0.95 and -0.9) and in all domain ($t \le 1$).

Table 2. Example 2 errors.

Dimension	$n=m=3$	$n=m=4$	$n=m=5$	$n=m=6$
$t \le -0.95$	0.006363	0.047758	0.019693	0.017333
$t \le -0.9$	0.012320	0.076309	0.036547	0.025828
$t \le 1$	0.770758	1.027078	1.024606	0.996752

Table 2 shows that the error decreases when the dimension increases in the points near of the initial point, -1, and in the whole domain.

Note that the dimension increment can produce an increment of the real solutions and the election of the correct solution can be complicated. For example, if the problem dimension is 6, two solutions are calculated. One of them has a small error near of the initial time point ($t=-1$) and after, its error increases hugely, while in the other solution, the error does not increase as much. Table 3 shows this error.

Table 3. Example 2 errors in the case $n=m=6$.

	$t \le -0.95$	$t \le -0.9$	$t \le -0.5$	$t \le 0$	$t \le 0.5$	$t \le 1$
first solution	0.01019	0.01810	0.57353	1.80015	2.40542	2.78428
second solution	0.01733	0.02582	0.37660	0.44356	0.47220	0.99675

6. EULER EQUATIONS APPLICATION

To finish, the collocation method is applied to the Euler equations. In this case, the system solution has two dimensions (T and u).

6.1. Algorithm

First, initial function (25) has to be calculated. For system (18), this function is

$$\delta(z,t) = g(z). \tag{59}$$

Using basis (37), collocations points (26), and the initial function, the algebraic system is

$$2\sum_{i,j=1}^{n,m} \alpha_{ij}^2(-1)^{i+j} = 0, \tag{60}$$

$$2A_3 \sum_{i,j=1}^{n,m} \alpha_{ij}^1 (-1)^{i+j} + 2 \sum_{i,j=1}^{n,m} \alpha_{ij}^2 (-1)^{i+j} = 0, \tag{61}$$

with $A_3 = R\gamma/(\gamma - 1)$, third element of the matrix of (9), by (32), and

$$\begin{aligned}
&\left(1 - \frac{1}{c_0}\left(g_1(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^1 B_{i,j}(x_i, z_j)\right)\right)\left(g_2'(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^2 \frac{\partial B_{i,j}(x_i, z_j)}{\partial z}\right) \\
&\quad - \frac{A_2}{c_0}\left(g_2(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^2 B_{i,j}(x_i, z_j)\right)\left(g_1'(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^1 \frac{\partial B_{i,j}(x_i, z_j)}{\partial z}\right) \\
&\quad + \left(g_2(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^2 B_{i,j}(x_i, z_j)\right)\left(\sum_{i,j=1}^{n,m} \alpha_{ij}^1 \frac{\partial B_{i,j}(x_i, z_j)}{\partial x}\right) \\
&\quad + \left(g_1(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^1 B_{i,j}(x_i, z_j)\right)\left(\sum_{i,j=1}^{n,m} \alpha_{ij}^2 \frac{\partial B_{i,j}(x_i, z_j)}{\partial x}\right) = 0
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
&\left(1 - \frac{1}{c_0}\left(g_1(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^1 B_{i,j}(x_i, z_j)\right)\right)\left(g_1'(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^1 \frac{\partial B_{i,j}(x_i, z_j)}{\partial z}\right) \\
&\quad - \frac{A_3}{c_0}\left(g_2'(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^2 \frac{\partial B_{i,j}(x_i, z_j)}{\partial z}\right) \\
&\quad + \left(g_2(z) + \sum_{i,j=1}^{n,m} \alpha_{ij}^2 B_{i,j}(x_i, z_j)\right)\left(\sum_{i,j=1}^{n,m} \alpha_{ij}^2 \frac{\partial B_{i,j}(x_i, z_j)}{\partial x}\right) \\
&\quad + A_3\left(\sum_{i,j=1}^{n,m} \alpha_{ij}^1 \frac{\partial B_{i,j}(x_i, z_j)}{\partial x}\right) = 0,
\end{aligned} \tag{63}$$

with $A_2 = \gamma - 1$ of (9), by (33).

If equation (60) is substituted in (61) and the constants are removed, equation (60) and (61) are equivalent to

$$\sum_{i,j=1}^{n,m} \alpha_{ij}^1 (-1)^{i+j} = 0, \qquad \sum_{i,j=1}^{n,m} \alpha_{ij}^2 (-1)^{i+j} = 0. \tag{64}$$

Summarizing, the algebraic system is defined for the equations (62)–(64).

If the equations of the points of the form $(x_i, -1)$ are studied thoroughly, it is observed that equations (62) are proportional to equations (63). This is showed as follows: if the points are substituted in the equation and as $g_1(-1) = T_0$, $g_2(-1) = 0$, $B_{ij}(x_i, -1) = 0$, and $\frac{\partial B_{ij}(x_i,-1)}{\partial x} = 0$, equations (62) and (63) are expressed as

$$\sum_{i,j=1}^{n,m} \alpha_{ij}^2 \frac{\partial B_{ij}(x_i, -1)}{\partial z} - \frac{A_2 T_0}{c_0} \sum_{i,j=1}^{n,m} \alpha_{ij}^1 \frac{\partial B_{ij}(x_i, -1)}{\partial z} = 0 \tag{65}$$

and

$$\sum_{i,j=1}^{n,m} \alpha_{ij}^1 \frac{\partial B_{ij}(x_i, -1)}{\partial z} - \frac{A_3}{c_0} \sum_{i,j=1}^{n,m} \alpha_{ij}^2 \frac{\partial B_{ij}(x_i, -1)}{\partial z} = 0. \tag{66}$$

Since $A_3 = \gamma R/(\gamma - 1)$, $A_2 = \gamma - 1$, and $c_0^2 = \gamma R T_0$, the following equality is obtained

$$\frac{A_3}{c_0} = \frac{\gamma R}{c_0(\gamma - 1)} = \frac{c_0}{T_0(\gamma - 1)} = \frac{c_0}{T_0 A_2}. \tag{67}$$

Therefore, equation (65) is equivalent to (66) and one of the two equations set has to be substituted. In this work, equation (66) has been replaced by the equations calculated on the points $(x_i, -0.5)$ (these equations were previously defined on the points $(x_i, -1)$). This election was used because more information is obtained if an intermediate point is chosen.

When the degree of each equation is obtained, it is easy to see that the equations coming from boundary points are linear and the other are quadratics. So, nm equations are linear and $2(n-1)(m-1)+m-1$ are quadratics.

Once the algebraic system has been obtained and in order to apply the homotopy continuation method, a partition $Z = \{Z_1, Z_2\}$, with

$$Z_1 = \{\alpha^1_{i,j}, \text{ for } 0 \le i,\ j \le n, m\} \qquad \text{and} \qquad Z_2 = \{\alpha^2_{i,j}, \text{ for } 0 \le i,\ j \le n, m\} \tag{68}$$

is a good one. Its degree matrix, if the points are ordered as (53), is

$$M_d = [M_1, M_2, \dots, M_{2m}]^{\mathsf{T}}, \tag{69}$$

where

$$\begin{aligned} M_1 &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \end{bmatrix}_{2\times n}, \\ M_i &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \end{bmatrix}_{2\times n}, \\ M_{m+1} &= \begin{bmatrix} 0 & 2 & \cdots & 2 \\ 1 & 2 & \cdots & 2 \end{bmatrix}_{2\times n}, \qquad \text{and} \\ M_{m+i} &= \begin{bmatrix} 1 & 2 & \cdots & 2 \\ 1 & 2 & \cdots & 2 \end{bmatrix}_{2\times n}, \qquad i = 2, \dots, m. \end{aligned}$$

Its Bezout number is less than $3^{(n-1)(m-1)}$, the Bezout number of (70).

$$M_d = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & 2 \\ 0 & 1 \\ 2 & 2 \\ \vdots & \vdots \\ 2 & 2 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} nm-1 \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} nm-1 \end{matrix} \tag{70}$$

6.2. Computational Results

Once the algorithm has been developed, several examples to validate the method are presented. To system (18), two cases of boundary conditions have been studied. So, the *casov*

$$\begin{aligned} u(0,t) &= A_v \sin(kt), \\ T(0,t) &= T_0 \left(1 + \frac{A_v \sin(kt)}{p_0}\right)^2, \end{aligned} \tag{71}$$

and *casop*

$$\begin{aligned} u(0,t) &= \frac{2c_0}{\gamma-1}\left[\left(\frac{p_0 + A_p \sin(kt)}{p_0}\right)^{(\gamma-1)/2\gamma} - 1\right], \\ T(0,t) &= T_0 \left(\frac{p_0 + A_p \sin(kt)}{p_0}\right)^{(\gamma-1)/\gamma}, \end{aligned} \tag{72}$$

have been imposed due to its importance in the analytical case [1]. To the case casop, the amplitudes 40000 and 60000 Pa have been studied, while to the case casov 150 m/s and 180 m/s amplitudes have been used.

The results are the following: if the dimension 3, 4, or 5 is used, the method obtains the initial solution, and, therefore, the linear solution [1] is obtained. Besides, to the dimension 5, other solution with small changes are obtained, although these changes are not significative.

When dimension 6 has been used, the collocation solution begins to generate nonlinearity effects. On the other hand, the number of studied cases increases due to the increment of Bezout number. For this dimension 6, the result of the casop with amplitude 60000 Pa is presented in Figure 1. In this figure can be seen the result for two space points, initial spatial and 0.1 m.

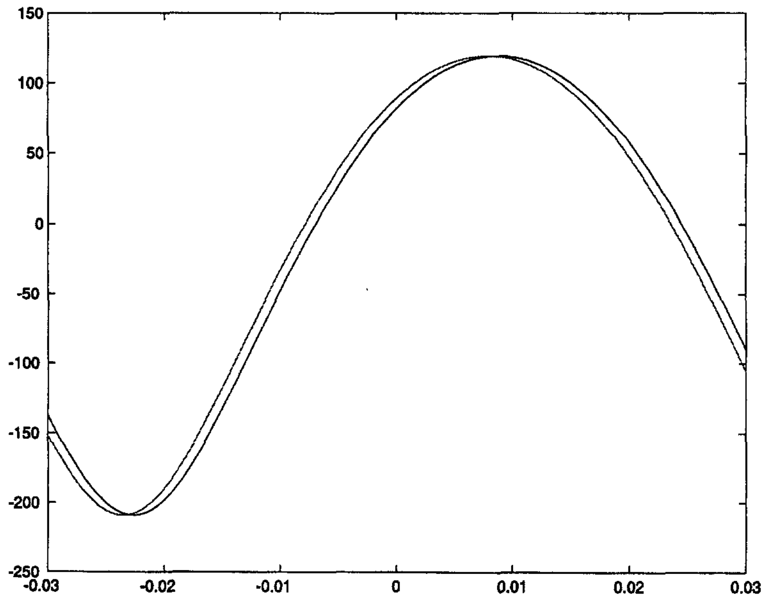

Figure 1. Velocity solution to casop.

7. CONCLUSIONS

In this article, a method to study quasilinear equations with polynomial matrix and initial or boundary conditions is presented. This method approximates the solution with the sum of a polynomial form with fixed degree, plus an initial approximation. This method has been applied to the Burgers equation and Euler equations, obtaining a good approximation.

The principal problem of the homotopy continuation methods for the collocation methods is that, as the homotopy methods searches all solutions, one has to be chosen and in the case that information about the solution is not known the method is not efficient. However, if some information of the solution is known, a stop condition has been imposed which can reduce the calculation time.

To conclude, we remark that the collocation polynomial basis used is the Chebyshev basis because this is analytical on the domain and this obtains the best polynomial approximation to the analytical function. If a less smooth solution is searched, a different basis can be used. Two examples of these basis are the Hermite polynomial basis and Splines, which are used in linear hyperbolic problem with good results [9,10] and they normally give a better-conditioned algebraic system.

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